Chapter 8. Converter Transfer Functions

Converter Transfer Functions

Design-oriented analysis

How to approach a real (and hence, complicated) system

Problems:

- Complicated derivations
- Long equations
- Algebra mistakes

Design objectives:

- Obtain physical insight which leads engineer to synthesis of a good design
- Obtain simple equations that can be inverted, so that element values can be chosen to obtain desired behavior. Equations that cannot be inverted are useless for design!

Design-oriented analysis is a structured approach to analysis, which attempts to avoid the above problems

Some elements of design-oriented analysis, discussed in this chapter

- Writing transfer functions in normalized form, to directly expose salient features
- Obtaining simple analytical expressions for asymptotes, corner frequencies, and other salient features, allows element values to be selected such that a given desired behavior is obtained
- Use of inverted poles and zeroes, to refer transfer function gains to the most important asymptote
- Analytical approximation of roots of high-order polynomials
- Graphical construction of Bode plots of transfer functions and polynomials, to

    avoid algebra mistakes

    approximate transfer functions

    obtain insight into origins of salient features

8.1. Review of Bode plots

Decibels

$$\| G \|_{\text{dB}} = 20 \log_{10}\left(\| G \| \right)$$

Decibels of quantities having units (impedance example): normalize before taking log

$$\| Z \|_{\text{dB}} = 20 \log_{10}\left(\frac{\| Z \|}{R_{base}} \right)$$

Table 8.1. Expressing magnitudes in decibels

Actual magnitude	*Magnitude in dB*
1/2	– 6dB
1	0 dB
2	6 dB
5 = 10/2	20 dB – 6 dB = 14 dB
10	20dB
$1000 = 10^3$	$3 \cdot 20$dB = 60 dB

5Ω is equivalent to 14dB with respect to a base impedance of R_{base} = 1Ω, also known as 14dBΩ.

60dBμA is a current 60dB greater than a base current of 1μA, or 1mA.

Bode plot of f^n

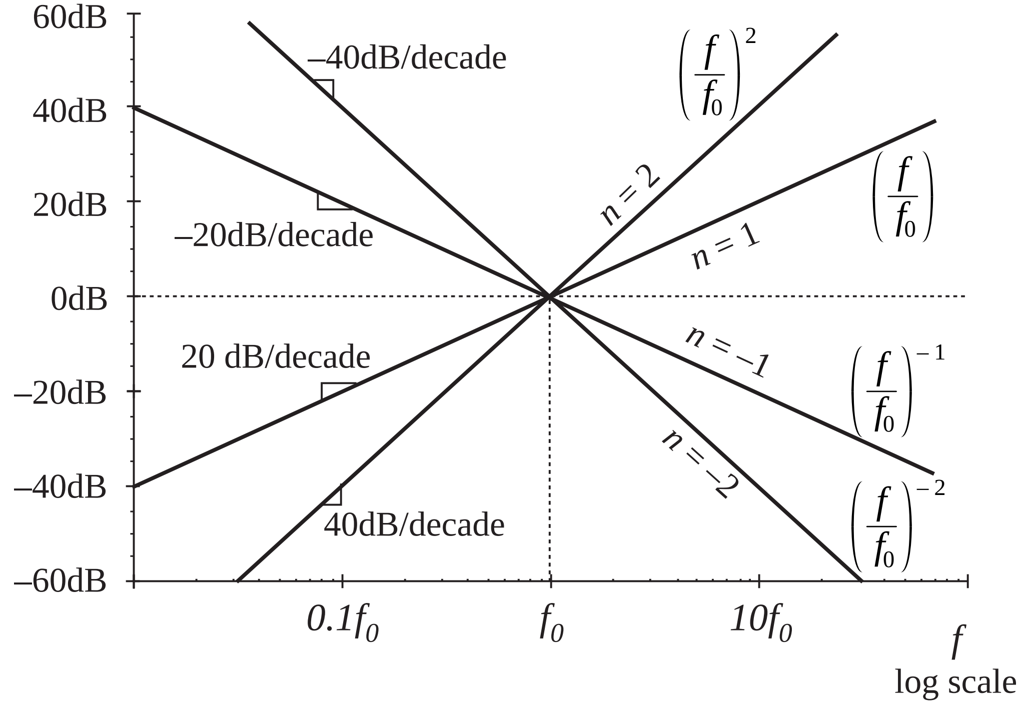
Bode plots are effectively log-log plots, which cause functions which vary as f^n to become linear plots. Given:

$$\left\| G \right\| = \left(\frac{f}{f_0} \right)^n$$

Magnitude in dB is

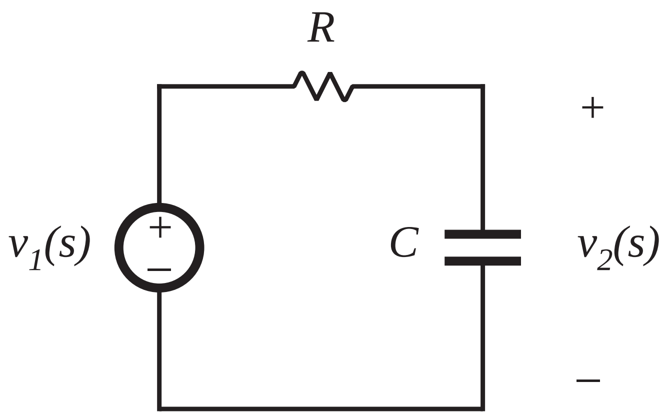
$$\left\| G \right\|_{\mathrm{dB}} = 20 \log_{10} \left(\frac{f}{f_0} \right)^n = 20n \log_{10} \left(\frac{f}{f_0} \right)$$

- Slope is $20n$ dB/decade
- Magnitude is 1, or 0dB, at frequency $f = f_0$



8.1.1. Single pole response

Simple R-C example



Transfer function is

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

Express as rational fraction:

$$G(s) = \frac{1}{1 + sRC}$$

This coincides with the normalized form

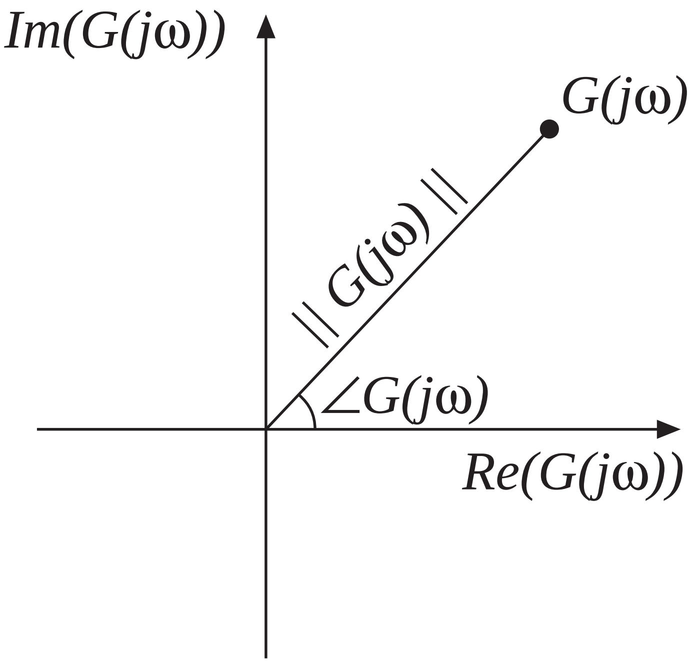
$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$

with $$\omega_0 = \frac{1}{RC}$$

$G(j\omega)$ and $\| G(j\omega) \|$

Let $s = j\omega$:

$$G(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_0}\right)} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$



Magnitude is

$$\left\| G(j\omega) \right\| = \sqrt{\left[\mathrm{Re}\,(G(j\omega))\right]^2 + \left[\mathrm{Im}\,(G(j\omega))\right]^2}$$
$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

Magnitude in dB:

$$\left\| G(j\omega) \right\|_{\mathrm{dB}} = -20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}\right)\ \mathrm{dB}$$

Asymptotic behavior: low frequency

For small frequency,
$\omega \ll \omega_0$ and $f \ll f_0$:

$$\left(\frac{\omega}{\omega_0} \right) \ll 1$$

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0} \right)^2}}$$

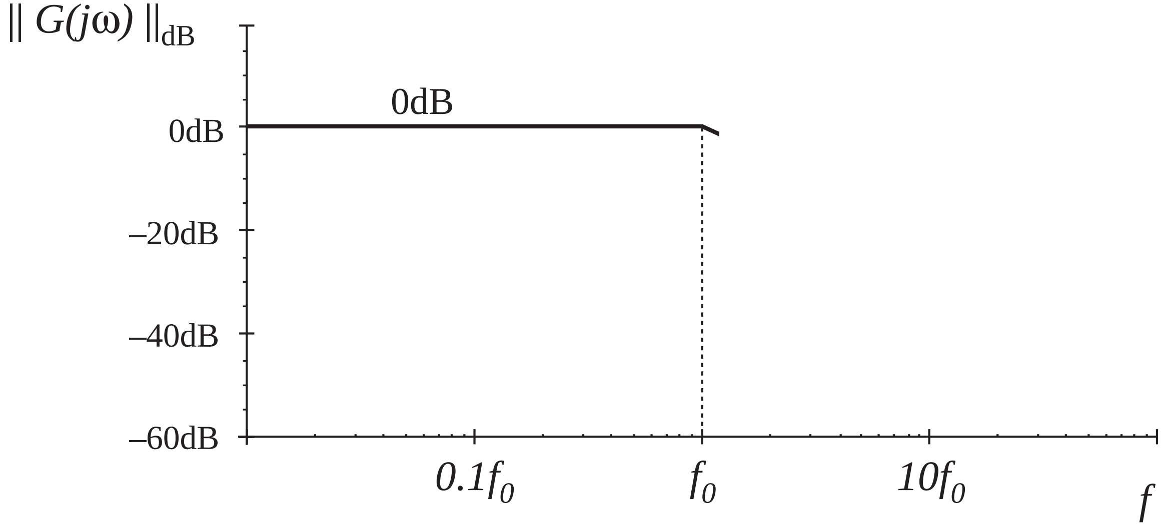
Then $\| G(j\omega) \|$ becomes

$$\left\| G(j\omega) \right\| \approx \frac{1}{\sqrt{1}} = 1$$

Or, in dB,

$$\left\| G(j\omega) \right\|_{dB} \approx 0\mathrm{dB}$$

This is the low-frequency asymptote of $\| G(j\omega) \|$



Asymptotic behavior: high frequency

For high frequency,
$\omega >> \omega_0$ and $f >> f_0$:

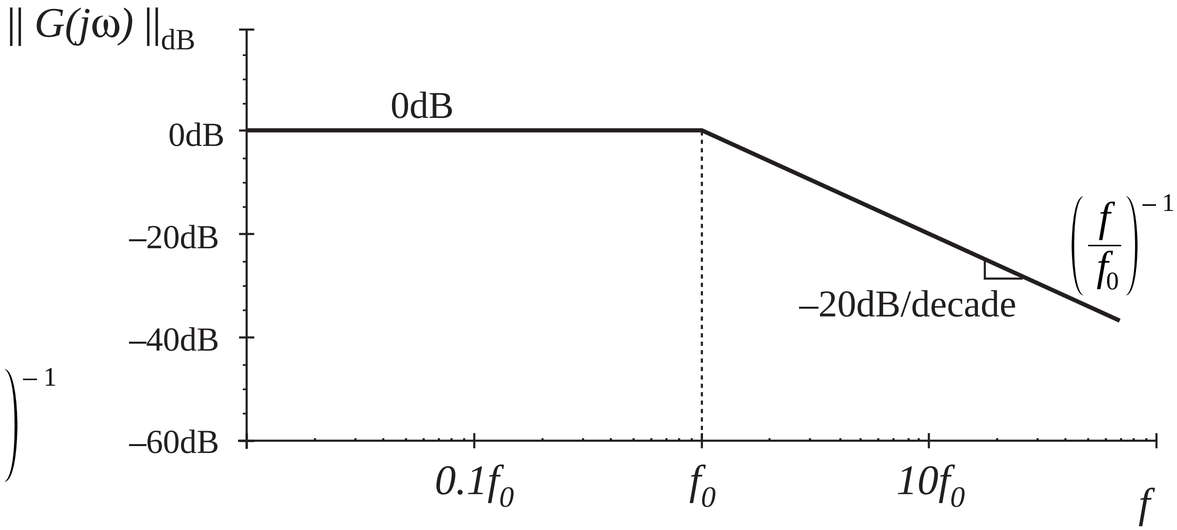
$$\left(\frac{\omega}{\omega_0}\right) >> 1$$

$$1 + \left(\frac{\omega}{\omega_0}\right)^2 \approx \left(\frac{\omega}{\omega_0}\right)^2$$

Then $\| G(j\omega) \|$ becomes

$$\left\| G(j\omega) \right\| \approx \frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2}} = \left(\frac{f}{f_0}\right)^{-1}$$

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$



The high-frequency asymptote of $\| G(j\omega) \|$ varies as f^{-1}. Hence, $n = -1$, and a straight-line asymptote having a slope of -20dB/decade is obtained. The asymptote has a value of 1 at $f = f_0$.

Deviation of exact curve near $f = f_0$

Evaluate exact magnitude:

at $f = f_0$:

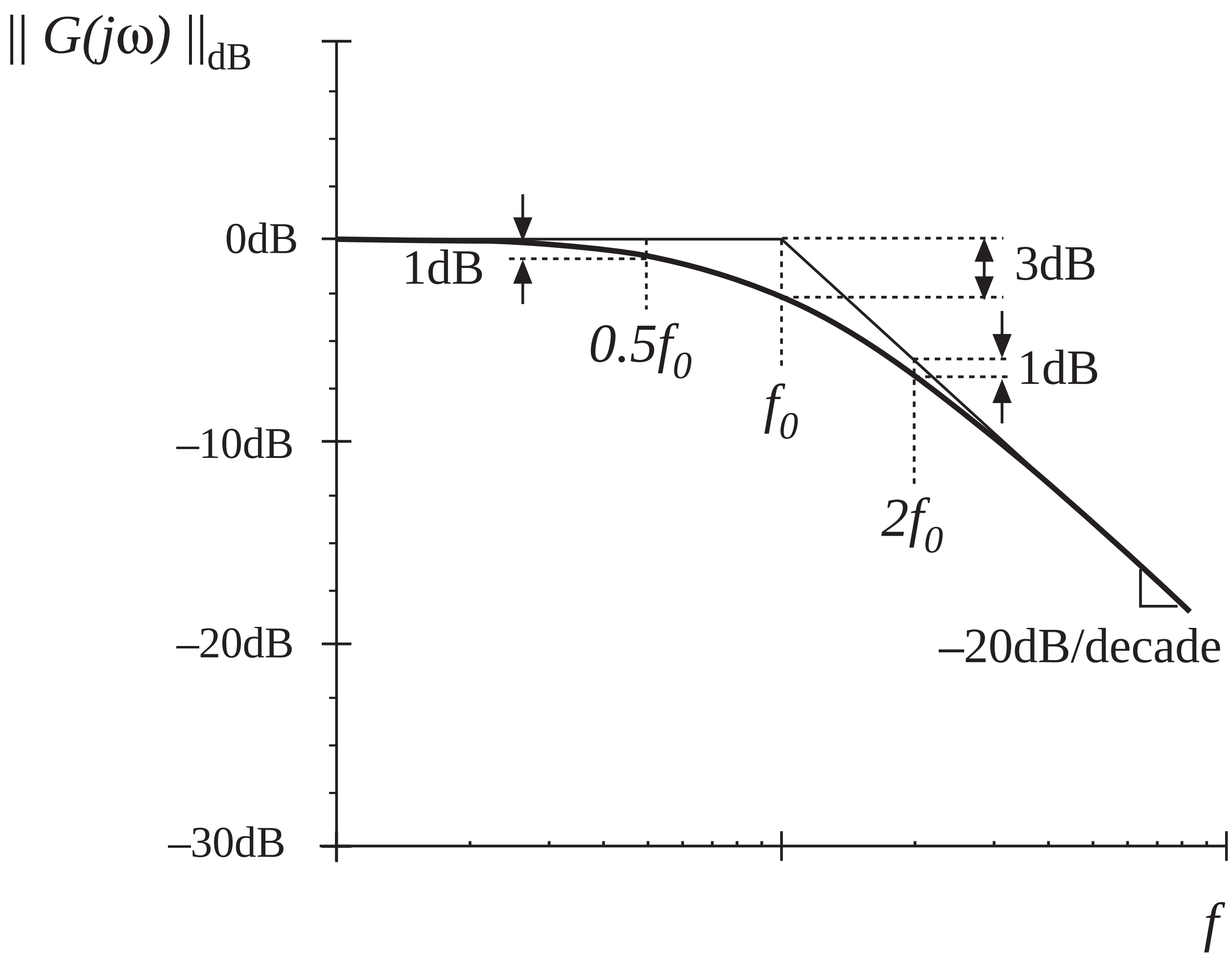
$$\left\| G(j\omega_0) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left\| G(j\omega_0) \right\|_{\text{dB}} = -20 \log_{10} \left(\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2} \right) \approx -3 \text{ dB}$$

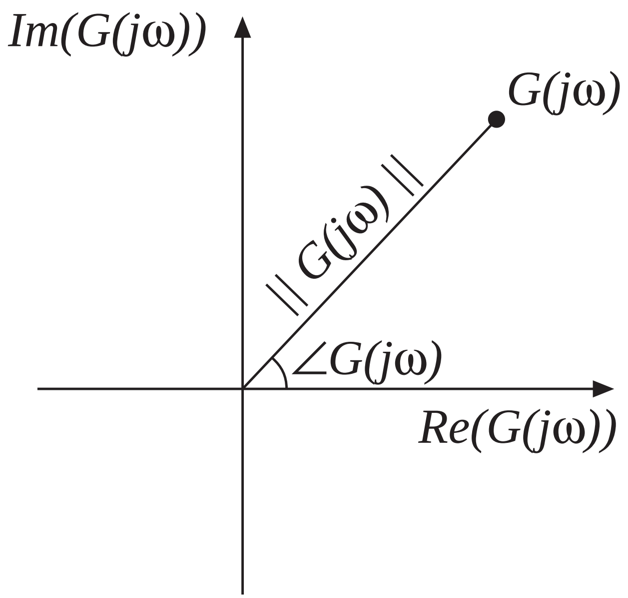
at $f = 0.5f_0$ *and* $2f_0$:

Similar arguments show that the exact curve lies 1dB below the asymptotes.

Summary: magnitude



Phase of $G(j\omega)$

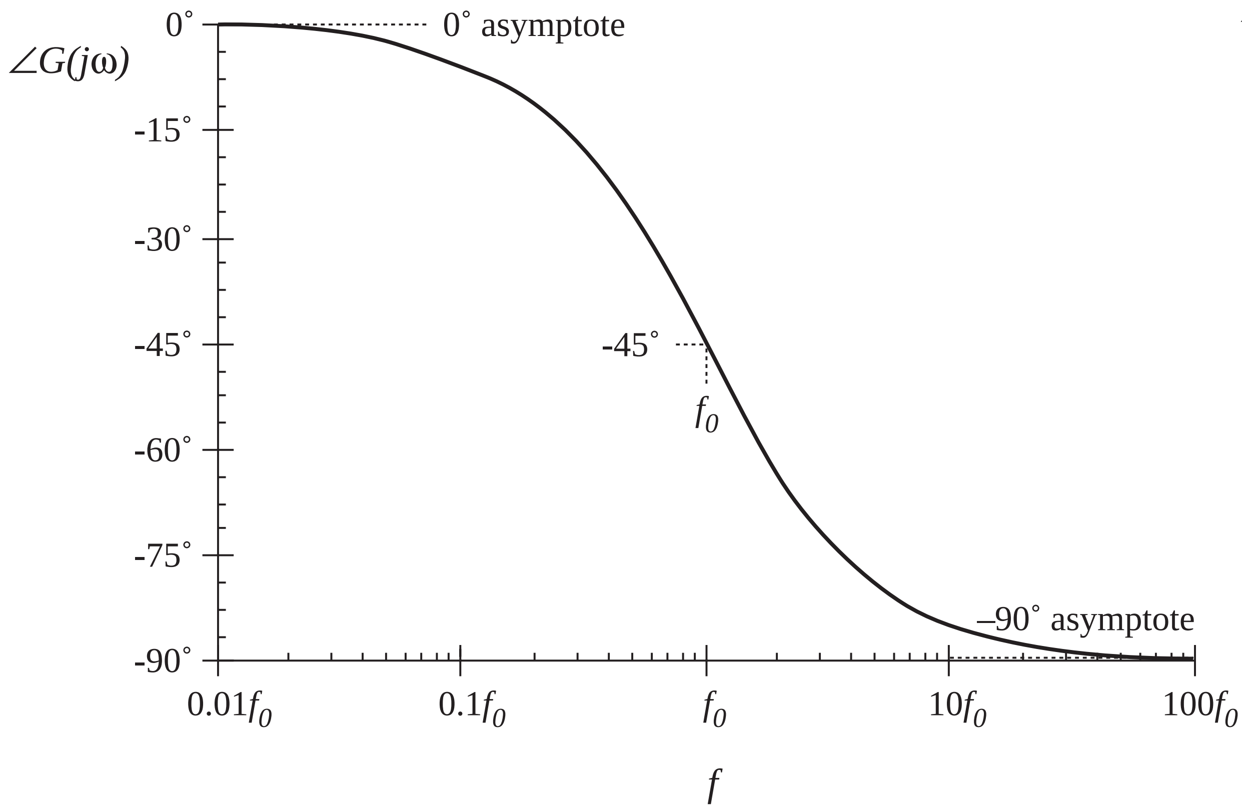


$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\left(G(j\omega)\right)}{\mathrm{Re}\left(G(j\omega)\right)}\right)$$

$$G(j\omega) = \frac{1}{\left(1 + j\,\frac{\omega}{\omega_0}\right)} = \frac{1 - j\,\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Phase of $G(j\omega)$



$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

ω	$\angle G(j\omega)$
0	0˚
ω_0	–45˚
∞	–90˚

Phase asymptotes

Low frequency: 0˚

High frequency: –90˚

Low- and high-frequency asymptotes do not intersect

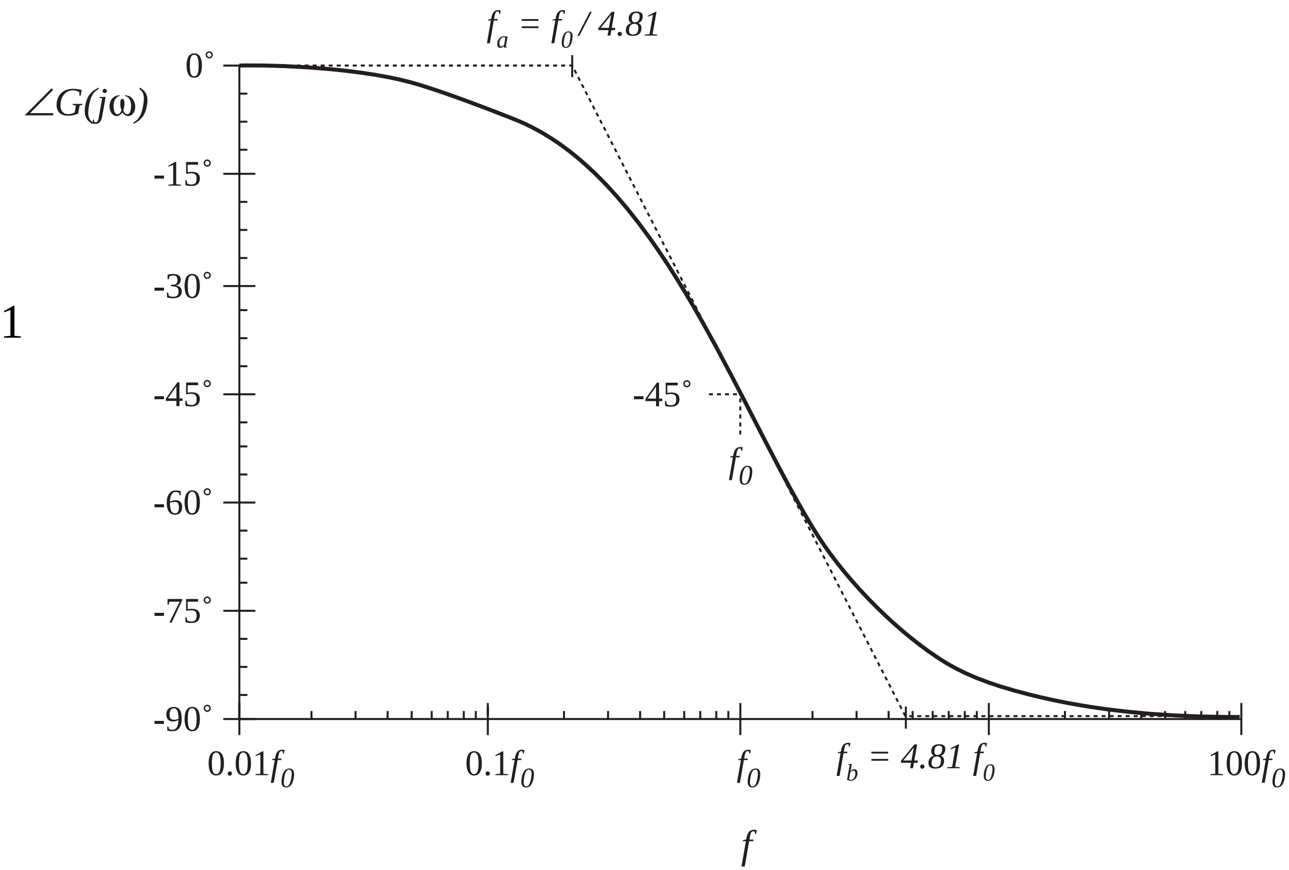
Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency f_0. One can show that the asymptotes then intersect at the break frequencies

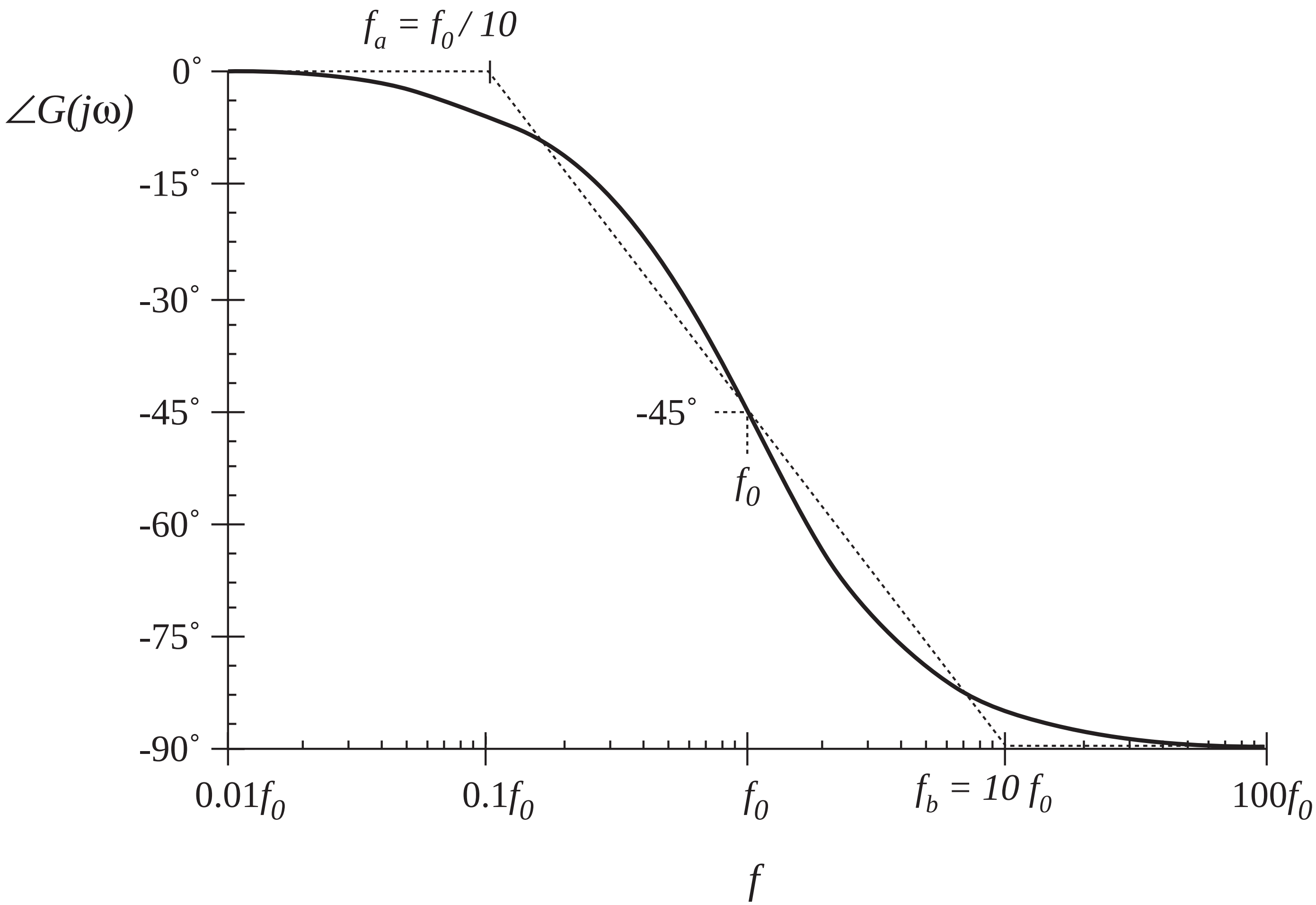
$$f_a = f_0\, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0\, e^{\pi/2} \approx 4.81\; f_0$$

Phase asymptotes

$$f_a = f_0\, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0\, e^{\pi/2} \approx 4.81\, f_0$$



Phase asymptotes: a simpler choice

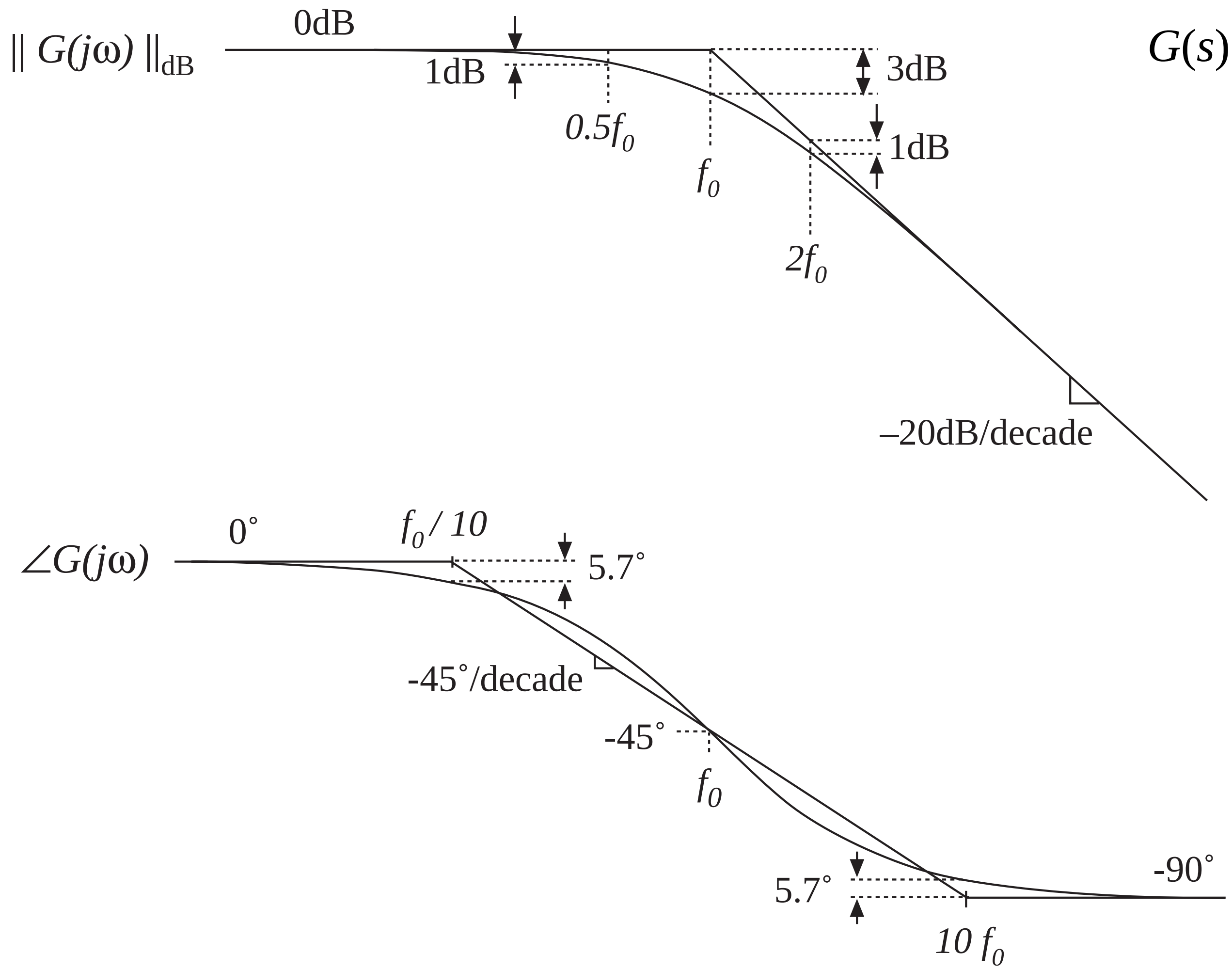


$f_a = f_0 / 10$

$f_b = 10\ f_0$

Summary: Bode plot of real pole

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$



8.1.2. Single zero response

Normalized form:

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$

Magnitude:

$$\left\| G(j\omega) \right\| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Use arguments similar to those used for the simple pole, to derive asymptotes:

0dB at low frequency, $\omega << \omega_0$

+20dB/decade slope at high frequency, $\omega >> \omega_0$

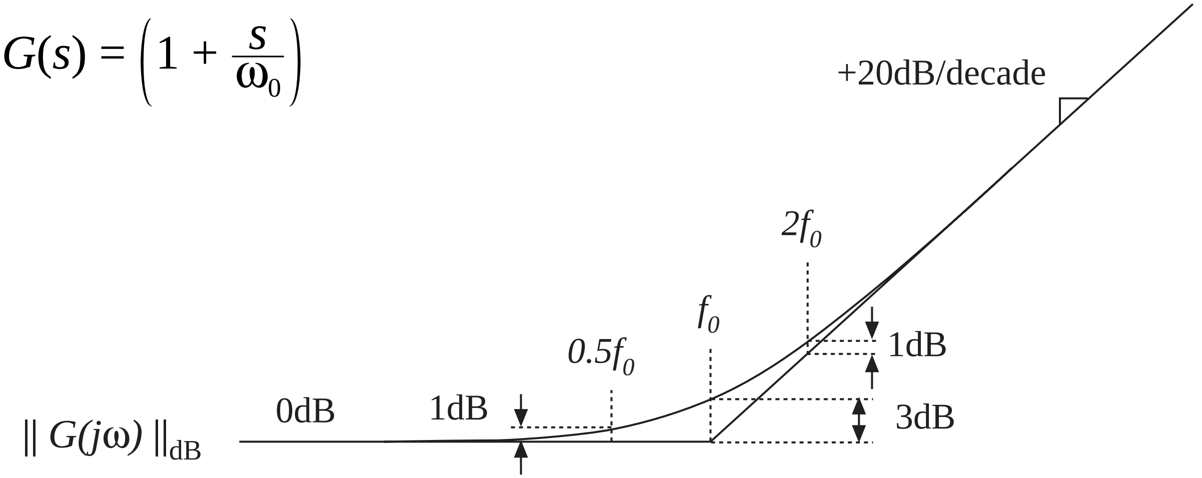
Phase:

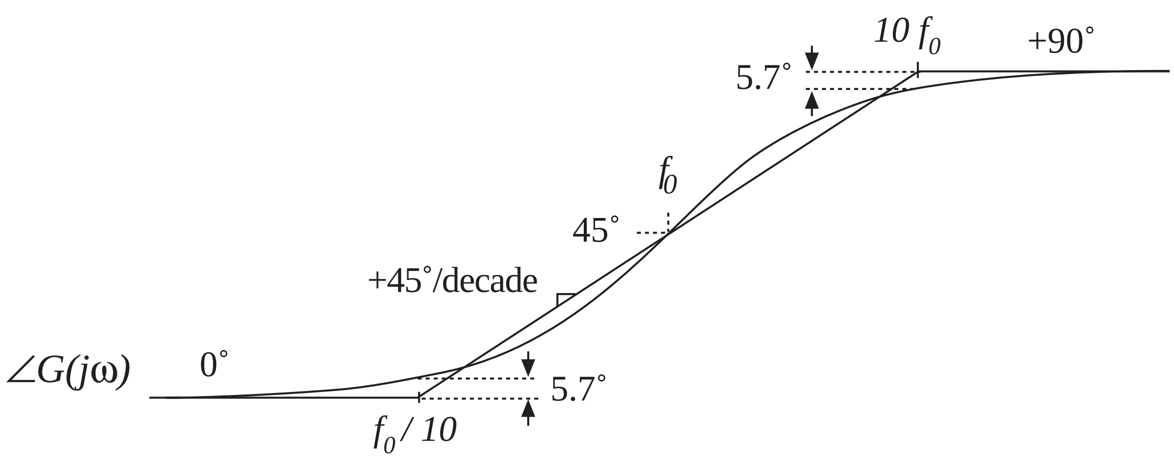
$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

—with the exception of a missing minus sign, same as simple pole

Summary: Bode plot, real zero

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$





8.1.3. Right half-plane zero

Normalized form:

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$

Magnitude:

$$\left\| G(j\omega) \right\| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

—same as conventional (left half-plane) zero. Hence, magnitude asymptotes are identical to those of LHP zero.

Phase:

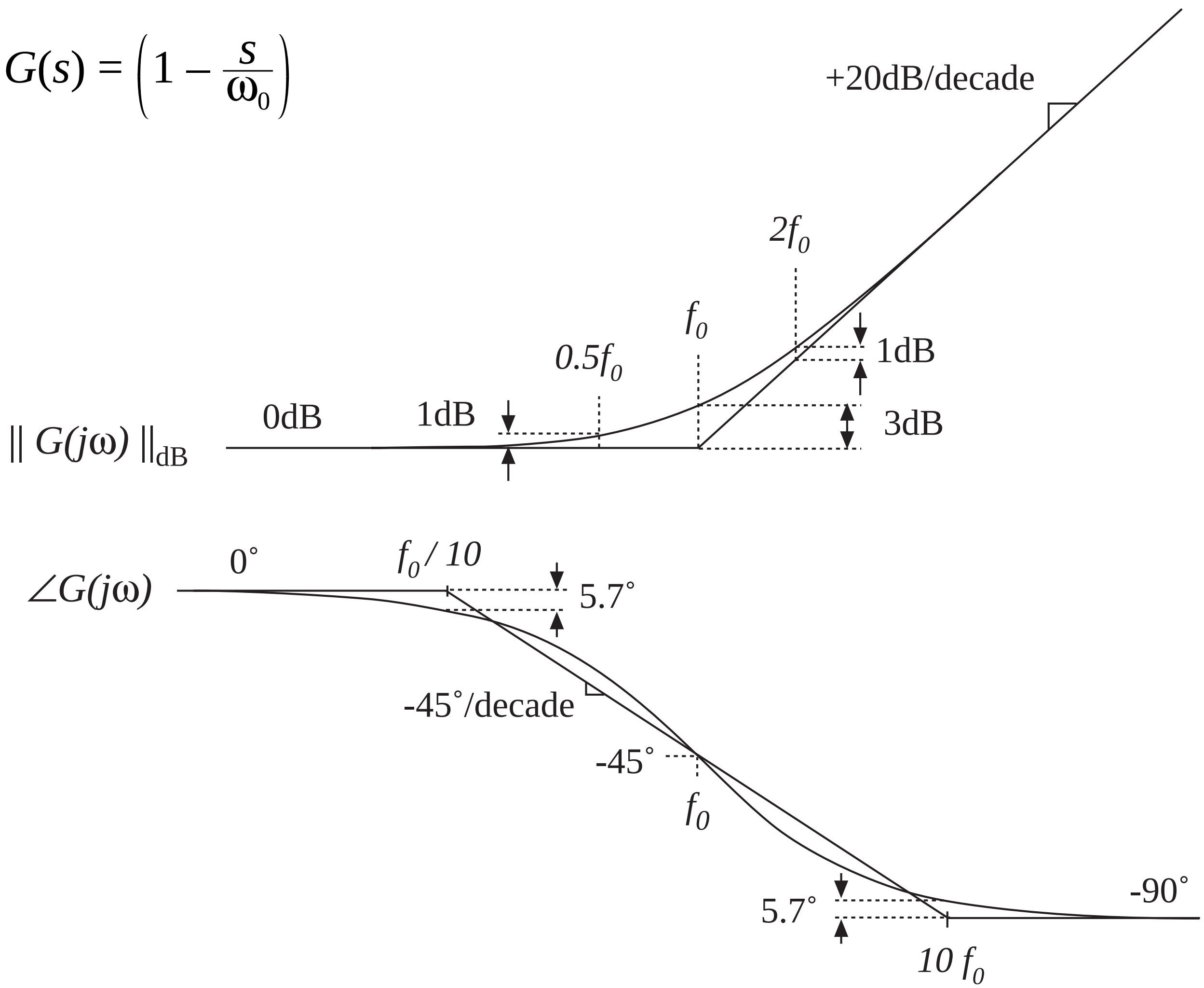
$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

—same as real pole.

The RHP zero exhibits the magnitude asymptotes of the LHP zero, and the phase asymptotes of the pole

Summary: Bode plot, RHP zero

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$



8.1.4. Frequency inversion

Reversal of frequency axis. A useful form when describing mid- or high-frequency flat asymptotes. Normalized form, inverted pole:

$$G(s) = \frac{1}{\left(1 + \frac{\omega_0}{s}\right)}$$

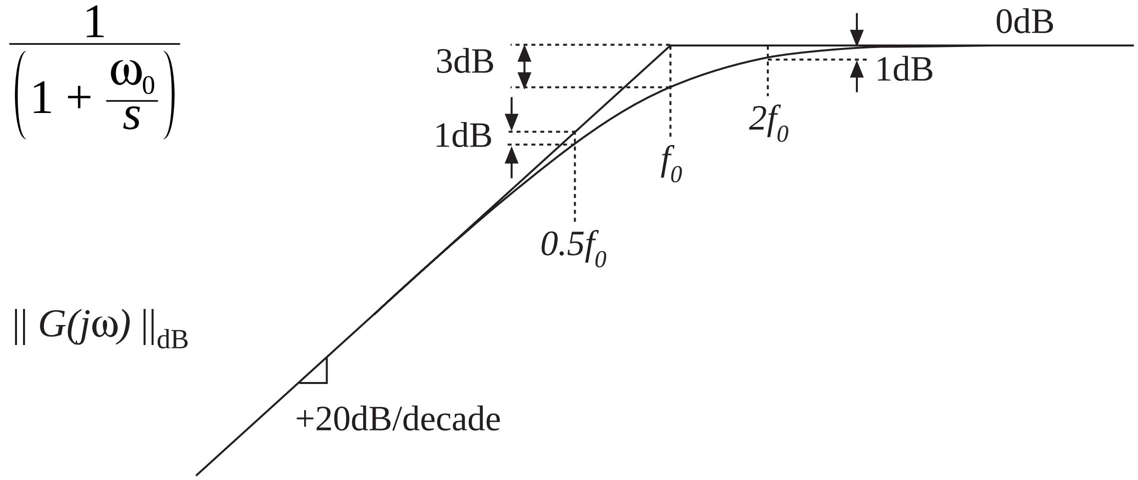
An algebraically equivalent form:

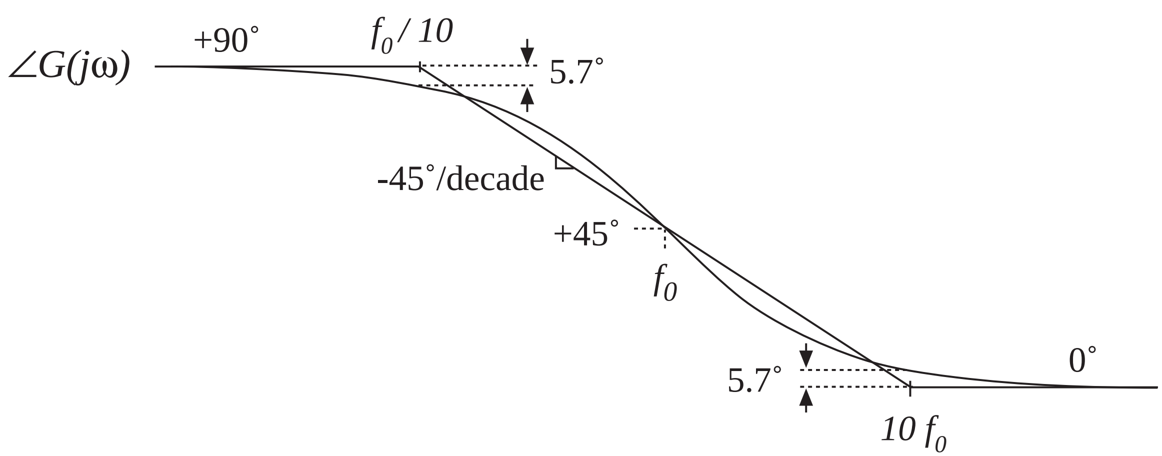
$$G(s) = \frac{\left(\frac{s}{\omega_0}\right)}{\left(1 + \frac{s}{\omega_0}\right)}$$

The inverted-pole format emphasizes the high-frequency gain.

Asymptotes, inverted pole

$$G(s) = \frac{1}{\left(1 + \frac{\omega_0}{s}\right)}$$





Inverted zero

Normalized form, inverted zero:

$$G(s) = \left(1 + \frac{\omega_0}{s}\right)$$

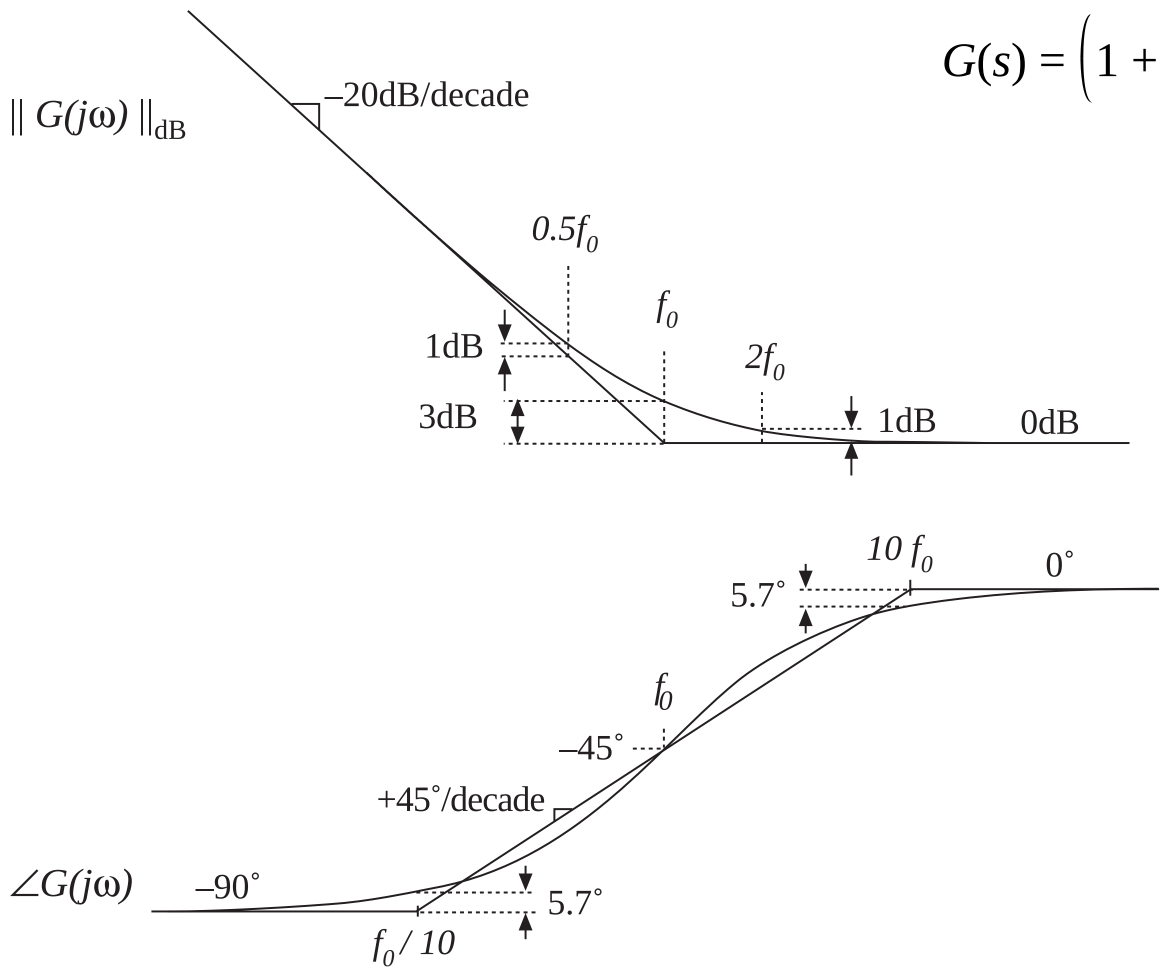
An algebraically equivalent form:

$$G(s) = \frac{\left(1 + \frac{s}{\omega_0}\right)}{\left(\frac{s}{\omega_0}\right)}$$

Again, the inverted-zero format emphasizes the high-frequency gain.

Asymptotes, inverted zero

$$G(s) = \left(1 + \frac{\omega_0}{s}\right)$$



8.1.5. Combinations

Suppose that we have constructed the Bode diagrams of two complex-values functions of frequency, $G_1(\omega)$ and $G_2(\omega)$. It is desired to construct the Bode diagram of the product, $G_3(\omega) = G_1(\omega)\, G_2(\omega)$.

Express the complex-valued functions in polar form:

$$G_1(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}$$

$$G_2(\omega) = R_2(\omega)\, e^{j\theta_2(\omega)}$$

$$G_3(\omega) = R_3(\omega)\, e^{j\theta_3(\omega)}$$

The product $G_3(\omega)$ can then be written

$$G_3(\omega) = G_1(\omega)\, G_2(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}\, R_2(\omega)\, e^{j\theta_2(\omega)}$$

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

Combinations

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

The composite phase is

$$\theta_3(\omega) = \theta_1(\omega) + \theta_2(\omega)$$

The composite magnitude is

$$R_3(\omega) = R_1(\omega)\, R_2(\omega)$$

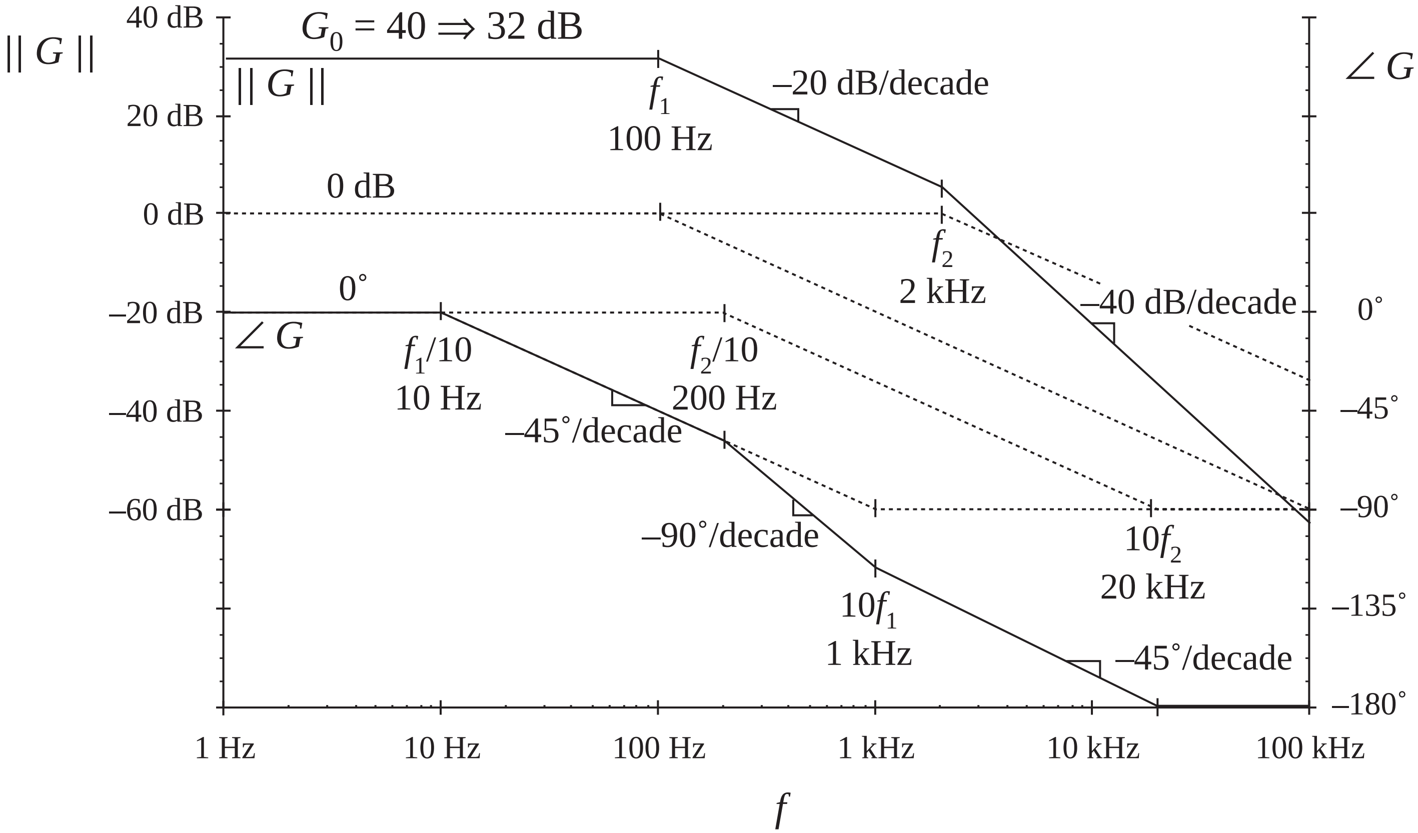
$$\left| R_3(\omega) \right|_{\mathrm{dB}} = \left| R_1(\omega) \right|_{\mathrm{dB}} + \left| R_2(\omega) \right|_{\mathrm{dB}}$$

Composite phase is sum of individual phases.

Composite magnitude, when expressed in dB, is sum of individual magnitudes.

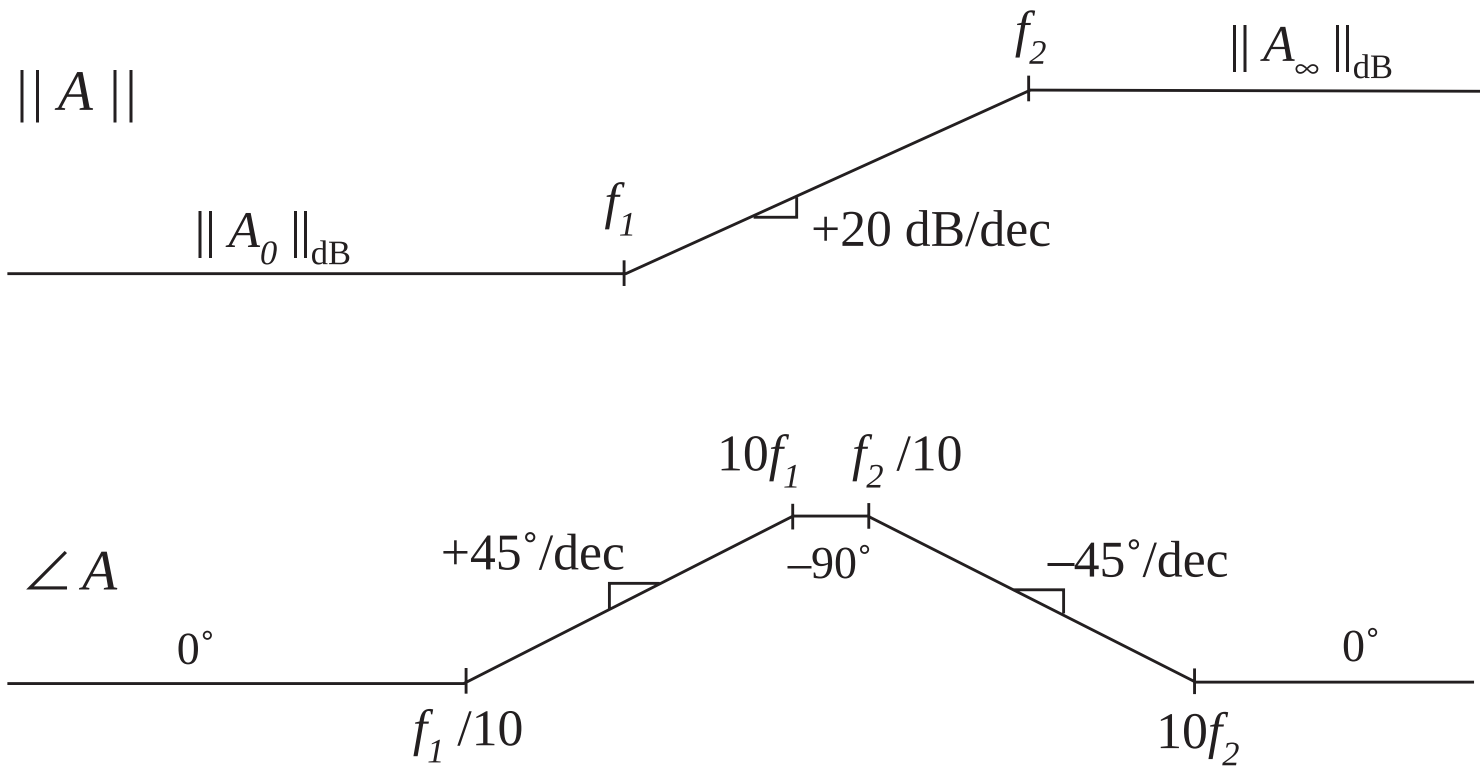
Example 1: $G(s) = \dfrac{G_0}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$

with $G_0 = 40 \Rightarrow 32$ dB, $f_1 = \omega_1/2\pi = 100$ Hz, $f_2 = \omega_2/2\pi = 2$ kHz



Example 2

Determine the transfer function *A(s)* corresponding to the following asymptotes:



Example 2, continued

One solution:

$$A(s) = A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)}$$

Analytical expressions for asymptotes:

For $f < f_1$

$$\left\| A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)} \right\|_{s = j\omega} = A_0 \frac{1}{1} = A_0$$

For $f_1 < f < f_2$

$$\left\| A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)} \right\|_{s = j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s = j\omega}}{1} = A_0 \frac{\omega}{\omega_1} = A_0 \frac{f}{f_1}$$

Example 2, continued

For $f > f_2$

$$\left\| A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)} \right\|_{s = j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s = j\omega}}{\left\| \frac{s}{\omega_2} \right\|_{s = j\omega}} = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{f_2}{f_1}$$

So the high-frequency asymptote is

$$A_\infty = A_0 \frac{f_2}{f_1}$$

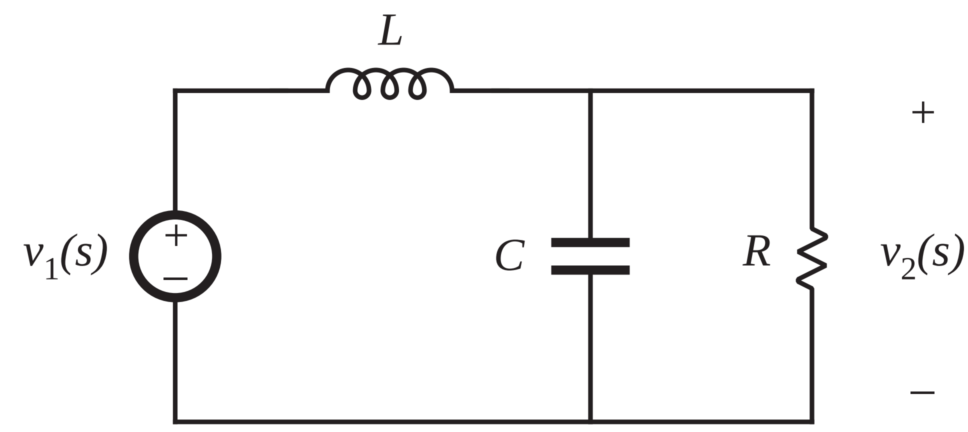
Another way to express $A(s)$: use inverted poles and zeroes, and express $A(s)$ directly in terms of A_∞

$$A(s) = A_\infty \frac{\left(1 + \frac{\omega_1}{s}\right)}{\left(1 + \frac{\omega_2}{s}\right)}$$

8.1.6 Quadratic pole response: resonance

Example

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$



Two-pole low-pass filter example

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

with $a_1 = L/R$ and $a_2 = LC$

How should we construct the Bode diagram?

Approach 1: factor denominator

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

We might factor the denominator using the quadratic formula, then construct Bode diagram as the combination of two real poles:

$$G(s) = \frac{1}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)} \qquad \text{with} \qquad s_1 = -\frac{a_1}{2a_2}\left[1 - \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

$$s_2 = -\frac{a_1}{2a_2}\left[1 + \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

- If $4a_2 \le a_1^2$, then the roots s_1 and s_2 are real. We can construct Bode diagram as the combination of two real poles.
- If $4a_2 > a_1^2$, then the roots are complex. In Section 8.1.1, the assumption was made that ω_0 is real; hence, the results of that section cannot be applied and we need to do some additional work.

Approach 2: Define a standard normalized form for the quadratic case

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of s are real and positive, then the parameters ζ, ω_0, and Q are also real and positive
- The parameters ζ, ω_0, and Q are found by equating the coefficients of s
- The parameter ω_0 is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter ζ is called the *damping factor.* ζ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter Q is called the *quality factor.* Q also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.

The Q-factor

In a second-order system, ζ and Q are related according to

$$Q = \frac{1}{2\zeta}$$

Q is a measure of the dissipation in the system. A more general definition of Q, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that Q has a simple interpretation in the Bode diagrams of second-order transfer functions.

Analytical expressions for f_0 and Q

Two-pole low-pass filter example: we found that

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Equate coefficients of like powers of s with the standard form

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Result:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

Magnitude asymptotes, quadratic form

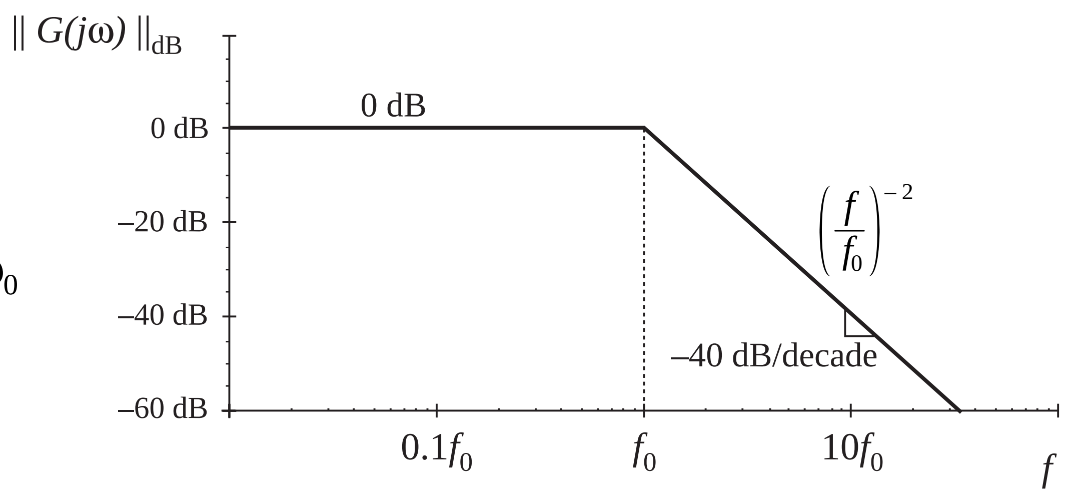
In the form $$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

let $s = j\omega$ and find magnitude: $$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

Asymptotes are

$$\left\| G \right\| \to 1 \quad \text{for } \omega \ll \omega_0$$

$$\left\| G \right\| \to \left(\frac{f}{f_0}\right)^{-2} \quad \text{for } \omega \gg \omega_0$$



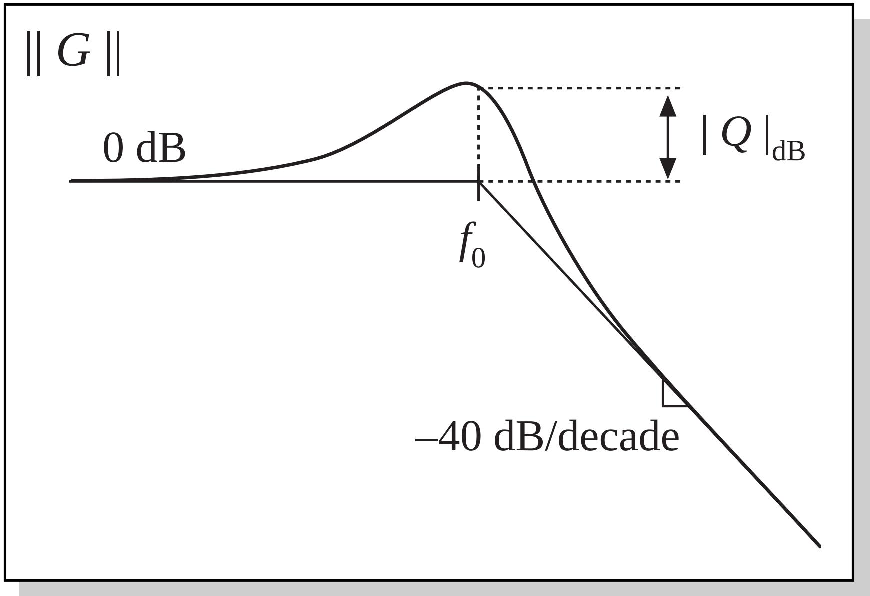
Deviation of exact curve from magnitude asymptotes

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

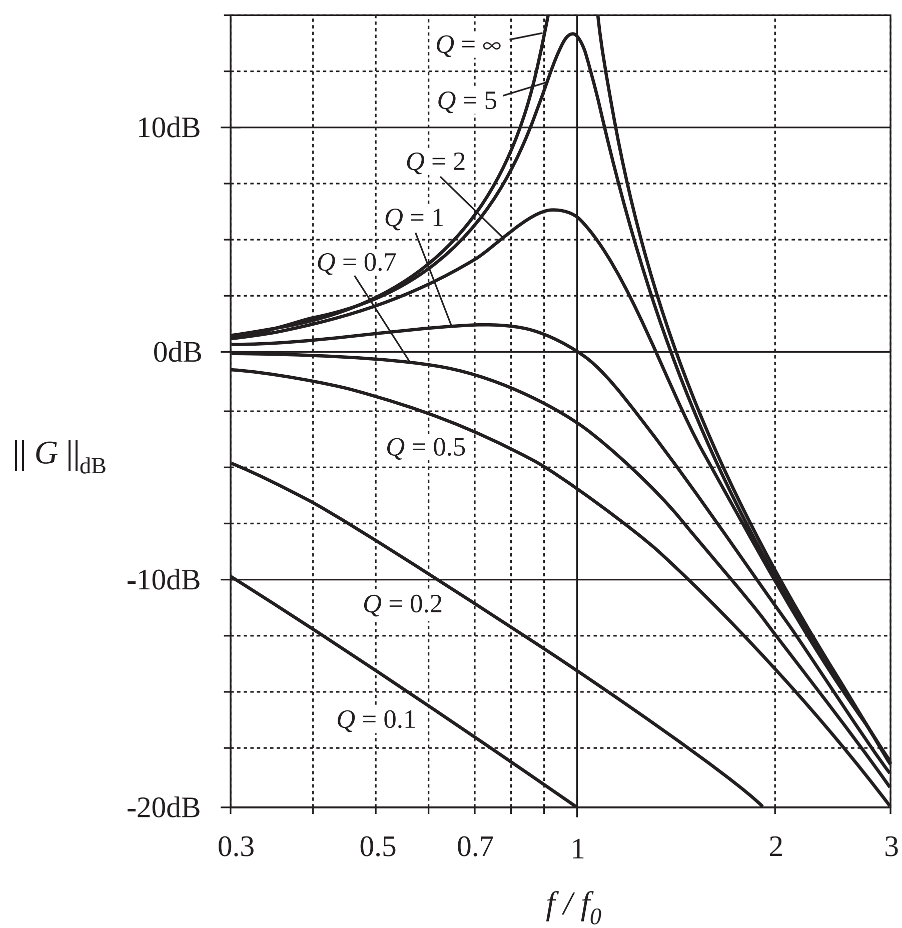
At $\omega = \omega_0$, the exact magnitude is

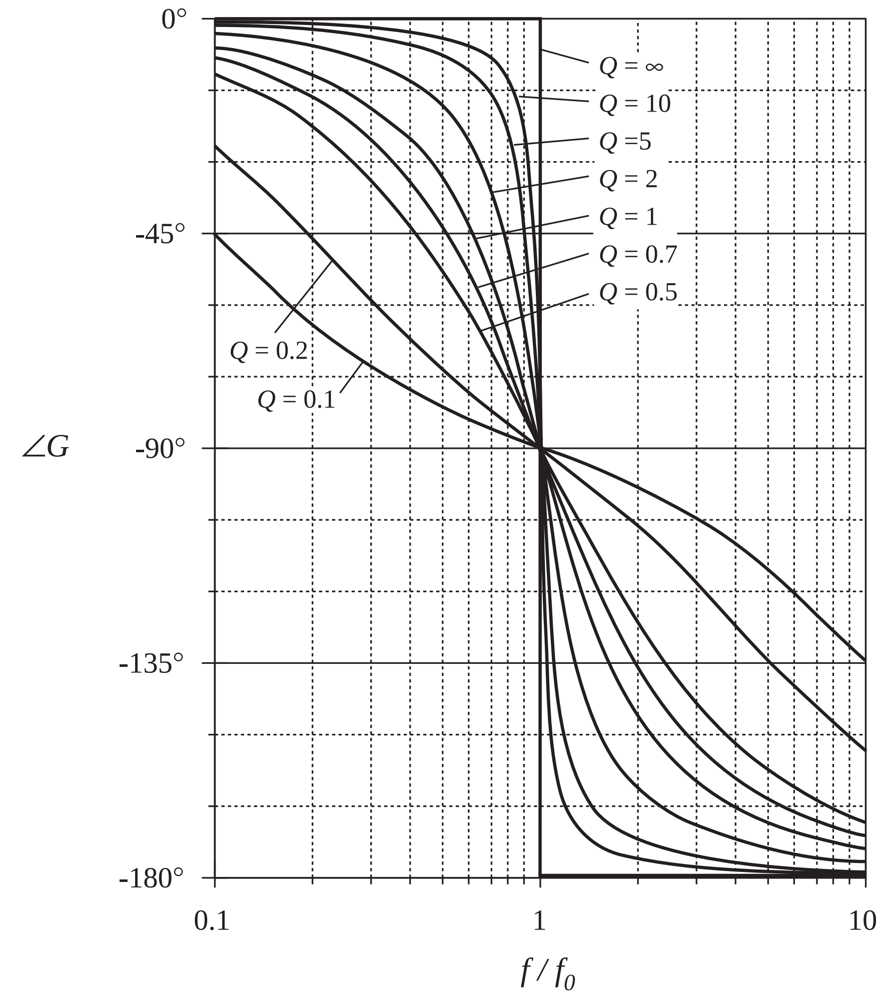
$\left\| G(j\omega_0) \right\| = Q$ or, in dB: $\left\| G(j\omega_0) \right\|_{\mathrm{dB}} = \left| Q \right|_{\mathrm{dB}}$

The exact curve has magnitude Q at $f = f_0$. The deviation of the exact curve from the asymptotes is $| Q |_{\mathrm{dB}}$



Two-pole response: exact curves





8.1.7. The low-Q approximation

Given a second-order denominator polynomial, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

When the roots are real, i.e., when Q < 0.5, then we can factor the denominator, and construct the Bode diagram using the asymptotes for real poles. We would then use the following normalized form:

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

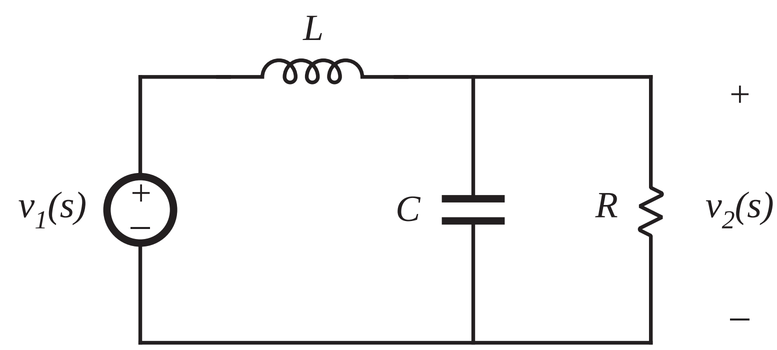
This is a particularly desirable approach when Q << 0.5, i.e., when the corner frequencies ω_1 and ω_2 are well separated.

An example

A problem with this procedure is the complexity of the quadratic formula used to find the corner frequencies.

R-L-C network example:

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$



Use quadratic formula to factor denominator. Corner frequencies are:

$$\omega_1,\, \omega_2 = \frac{L/R \pm \sqrt{\left(L/R\right)^2 - 4\,LC}}{2\,LC}$$

Factoring the denominator

$$\omega_1, \omega_2 = \frac{L/R \pm \sqrt{\left(L/R\right)^2 - 4\,LC}}{2\,LC}$$

This complicated expression yields little insight into how the corner frequencies ω_1 and ω_2 depend on R, L, and C.

When the corner frequencies are well separated in value, it can be shown that they are given by the much simpler (approximate) expressions

$$\omega_1 \approx \frac{R}{L}, \qquad \omega_2 \approx \frac{1}{RC}$$

ω_1 is then independent of C, and ω_2 is independent of L.

These simpler expressions can be derived via the Low-Q Approximation.

Derivation of the Low-Q Approximation

Given

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Use quadratic formula to express corner frequencies ω_1 and ω_2 in terms of Q and ω_0 as:

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2} \qquad \omega_2 = \frac{\omega_0}{Q} \frac{1 + \sqrt{1 - 4Q^2}}{2}$$

Corner frequency ω_2

$$\omega_2 = \frac{\omega_0}{Q}\,\frac{1 + \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

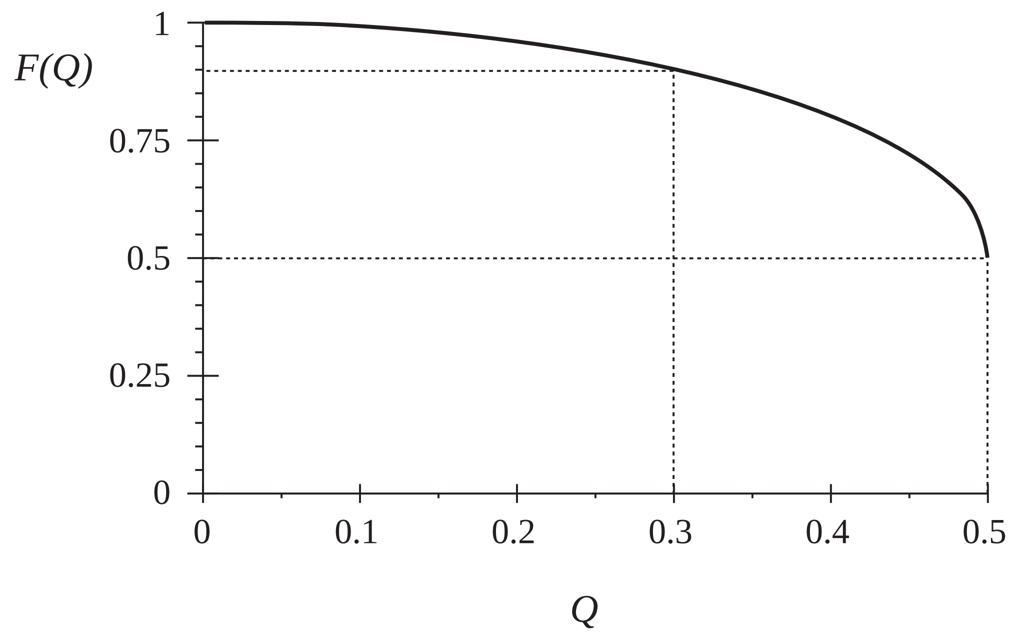
$$\omega_2 = \frac{\omega_0}{Q}\,F(Q)$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

For small Q, $F(Q)$ tends to 1. We then obtain

$$\omega_2 \approx \frac{\omega_0}{Q} \qquad \text{for } \ Q << \frac{1}{2}$$



For Q < 0.3, the approximation $F(Q)$ = 1 is within 10% of the exact value.

Corner frequency ω_1

$$\omega_1 = \frac{\omega_0}{Q} \frac{1 - \sqrt{1 - 4Q^2}}{2}$$

can be written in the form

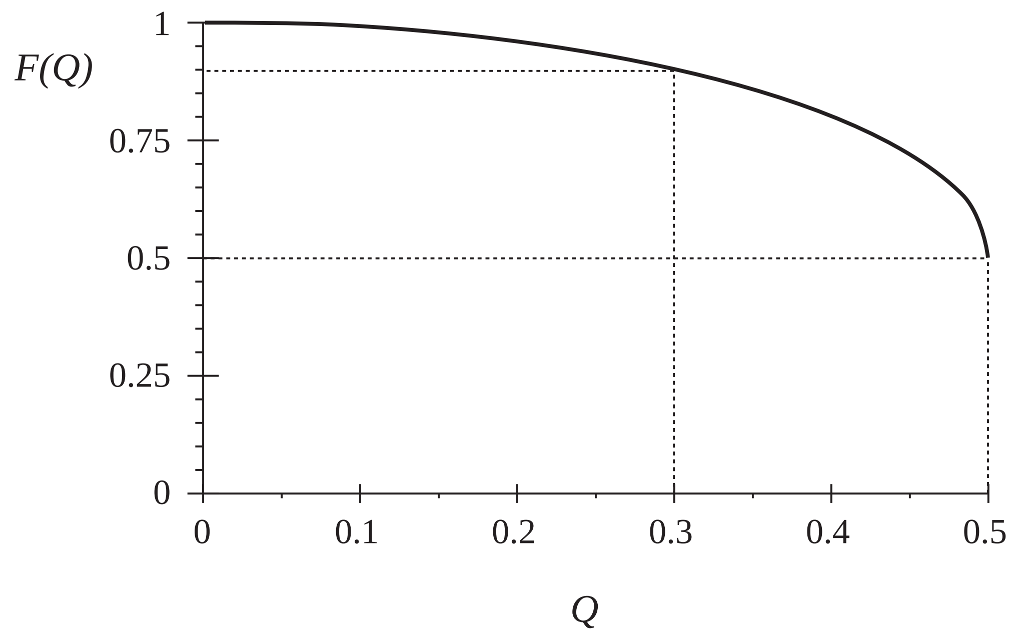
$$\omega_1 = \frac{Q\,\omega_0}{F(Q)}$$

where

$$F(Q) = \frac{1}{2}\left(1 + \sqrt{1 - 4Q^2}\right)$$

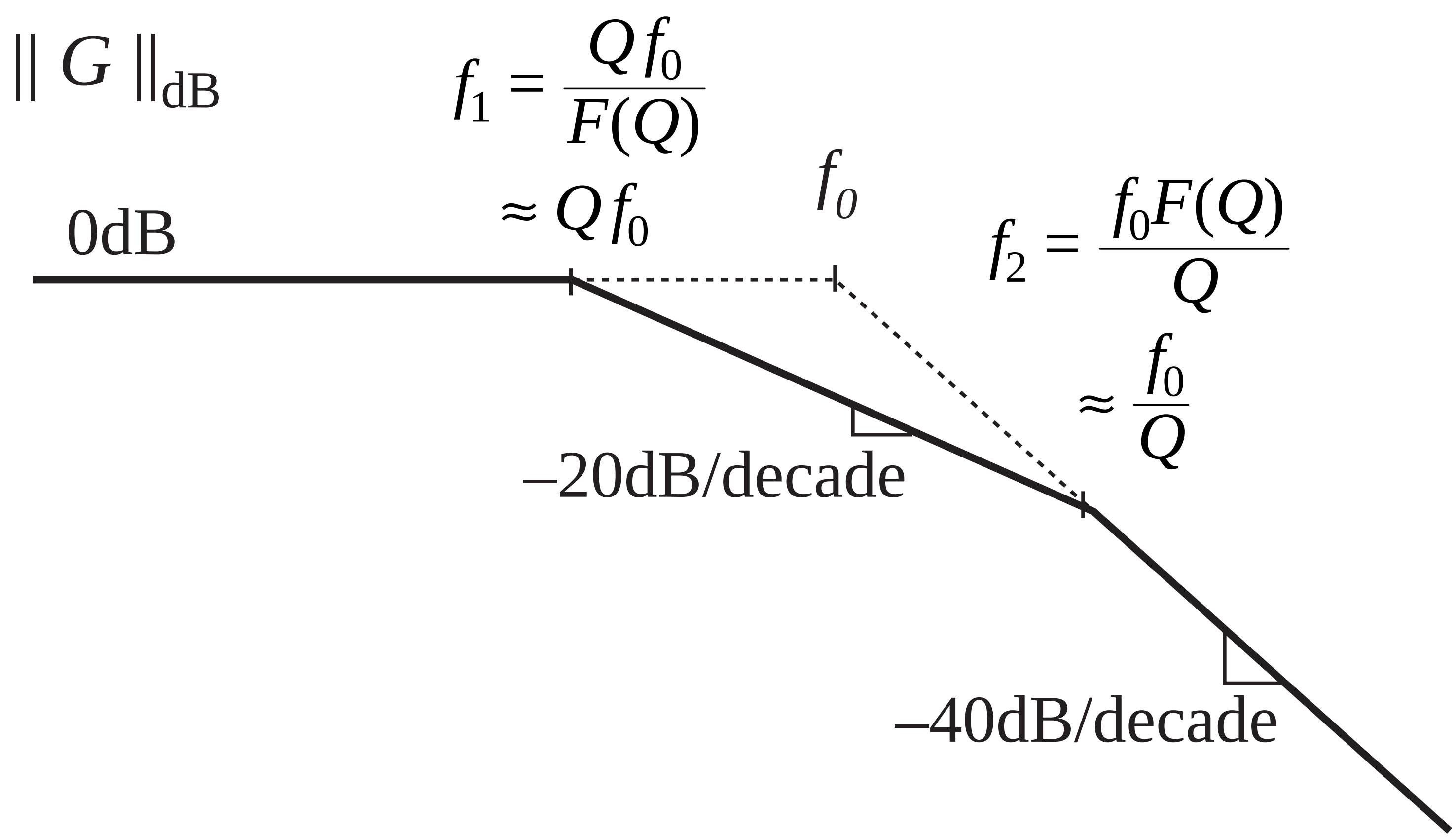
For small Q, $F(Q)$ tends to 1.
We then obtain

$$\omega_1 \approx Q\,\omega_0 \qquad \text{for } \; Q \ll \frac{1}{2}$$



For Q < 0.3, the approximation $F(Q)$ = 1 is within 10% of the exact value.

The Low-Q Approximation



R-L-C Example

For the previous example:

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

Use of the Low-Q Approximation leads to

$$\omega_1 \approx Q\,\omega_0 = R\sqrt{\frac{C}{L}}\,\frac{1}{\sqrt{LC}} = \frac{R}{L}$$

$$\omega_2 \approx \frac{\omega_0}{Q} = \frac{1}{\sqrt{LC}}\,\frac{1}{R\sqrt{\frac{C}{L}}} = \frac{1}{RC}$$

8.2. Analysis of converter transfer functions

8.2.1. Example: transfer functions of the buck-boost converter

8.2.2. Transfer functions of some basic CCM converters

8.2.3. Physical origins of the right half-plane zero in converters

8.2.1. Example: transfer functions of the buck-boost converter

Small-signal ac equations of the buck-boost converter, derived in section 7.2:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

Definition of transfer functions

The converter contains two inputs, $\hat{d}(s)$ and $\hat{v}_g(s)$ and one output, $\hat{v}(s)$

Hence, the ac output voltage variations can be expressed as the superposition of terms arising from the two inputs:

$$\hat{v}(s) = G_{vd}(s)\,\hat{d}(s) + G_{vg}(s)\,\hat{v}_g(s)$$

The control-to-output and line-to-output transfer functions can be defined as

$$G_{vd}(s) = \left.\frac{\hat{v}(s)}{\hat{d}(s)}\right|_{\hat{v}_g(s)\,=\,0} \quad \text{and} \quad G_{vg}(s) = \left.\frac{\hat{v}(s)}{\hat{v}_g(s)}\right|_{\hat{d}(s)\,=\,0}$$

Derivation of transfer functions

Algebraic approach

Take Laplace transform of converter equations, letting initial conditions be zero:

$$sL\hat{i}(s) = D\hat{v}_g(s) + D'\hat{v}(s) + \left(V_g - V\right)\hat{d}(s)$$

$$sC\hat{v}(s) = -D'\hat{i}(s) - \frac{\hat{v}(s)}{R} + I\hat{d}(s)$$

Eliminate $\hat{i}(s)$, and solve for $\hat{v}(s)$

$$\hat{i}(s) = \frac{D\hat{v}_g(s) + D'\hat{v}(s) + \left(V_g - V\right)\hat{d}(s)}{sL}$$

Derivation of transfer functions

$$sC\hat{v}(s) = -\frac{D'}{sL}\left(D\hat{v}_g(s) + D'\hat{v}(s) + \left(V_g - V\right)\hat{d}(s)\right) - \frac{\hat{v}(s)}{R} + I\hat{d}(s)$$

$$\hat{v}(s) = \frac{-DD'}{D'^2 + s\frac{L}{R} + s^2 LC}\,\hat{v}_g(s) - \frac{V_g - V - s\,LI}{D'^2 + s\frac{L}{R} + s^2 LC}\,\hat{d}(s)$$

write in normalized form:

$$\hat{v}(s) = \left(-\frac{D}{D'}\right)\frac{1}{1 + s\frac{L}{D'^2 R} + s^2\frac{LC}{D'^2}}\,\hat{v}_g(s) - \left(\frac{V_g - V}{D'^2}\right)\frac{\left(1 - s\frac{LI}{V_g - V}\right)}{1 + s\frac{L}{D'^2 R} + s^2\frac{LC}{D'^2}}\,\hat{d}(s)$$

Derivation of transfer functions

Hence, the line-to-output transfer function is

$$G_{vg}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s) = 0} = \left(- \frac{D}{D'} \right) \frac{1}{1 + s \frac{L}{D'^2 R} + s^2 \frac{LC}{D'^2}}$$

which is of the following standard form:

$$G_{vg}(s) = G_{g0} \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0} \right)^2}$$

Salient features of the line-to-output transfer function

Equate standard form to derived transfer function, to determine expressions for the salient features:

$$G_{g0} = -\frac{D}{D'}$$

$$\frac{1}{\omega_0^2} = \frac{LC}{D'^2} \qquad \omega_0 = \frac{D'}{\sqrt{LC}}$$

$$\frac{1}{Q\omega_0} = \frac{L}{D'^2 R} \qquad Q = D'R\sqrt{\frac{C}{L}}$$

Control-to-output transfer function

$$G_{vd}(s) = \left.\frac{\hat{v}(s)}{\hat{d}(s)}\right|_{\hat{v}_g(s)=0} = \left(-\frac{V_g - V}{D'^2}\right) \frac{\left(1 - s\,\frac{LI}{V_g - V}\right)}{\left(1 + s\,\frac{L}{D'^2 R} + s^2\,\frac{LC}{D'^2}\right)}$$

Standard form:

$$G_{vd}(s) = G_{d0}\,\frac{\left(1 - \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2\right)}$$

Salient features of control-to-output transfer function

$$G_{d0} = -\frac{V_g - V}{D'^2} = -\frac{V_g}{D'^3} = \frac{V}{D\,D'^2}$$

$$\omega_z = \frac{V_g - V}{L\,I} = \frac{D'\,R}{D\,L} \qquad \text{(RHP)}$$

$$\omega_0 = \frac{D'}{\sqrt{LC}}$$

$$Q = D'R\sqrt{\frac{C}{L}}$$

— Simplified using the dc relations: $$V = -\frac{D}{D'}V_g$$ $$I = -\frac{V}{D'\,R}$$

Plug in numerical values

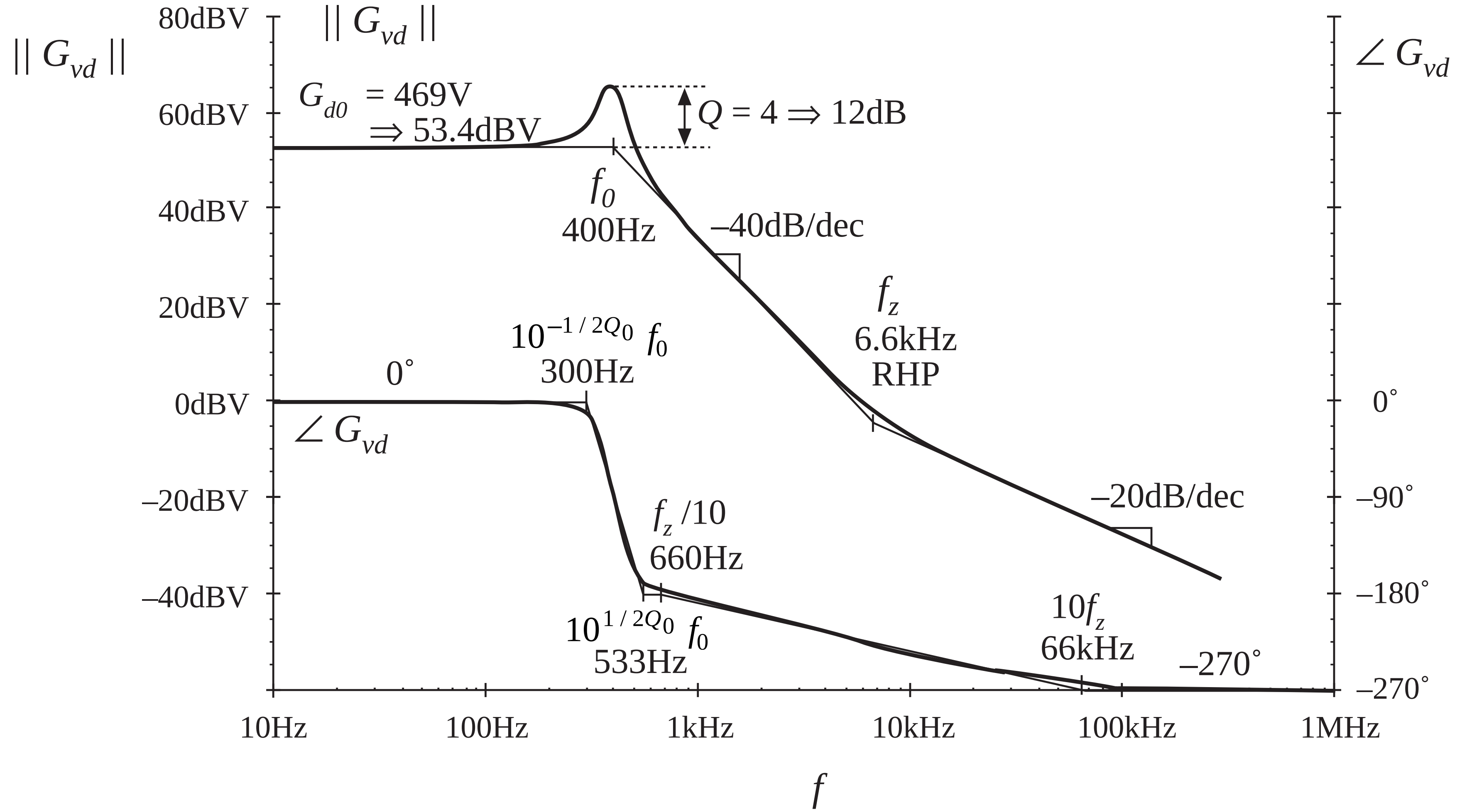
Suppose we are given the following numerical values:

$$
\begin{aligned}
D &= 0.6 \\
R &= 10\Omega \\
V_g &= 30\text{V} \\
L &= 160\mu\text{H} \\
C &= 160\mu\text{F}
\end{aligned}
$$

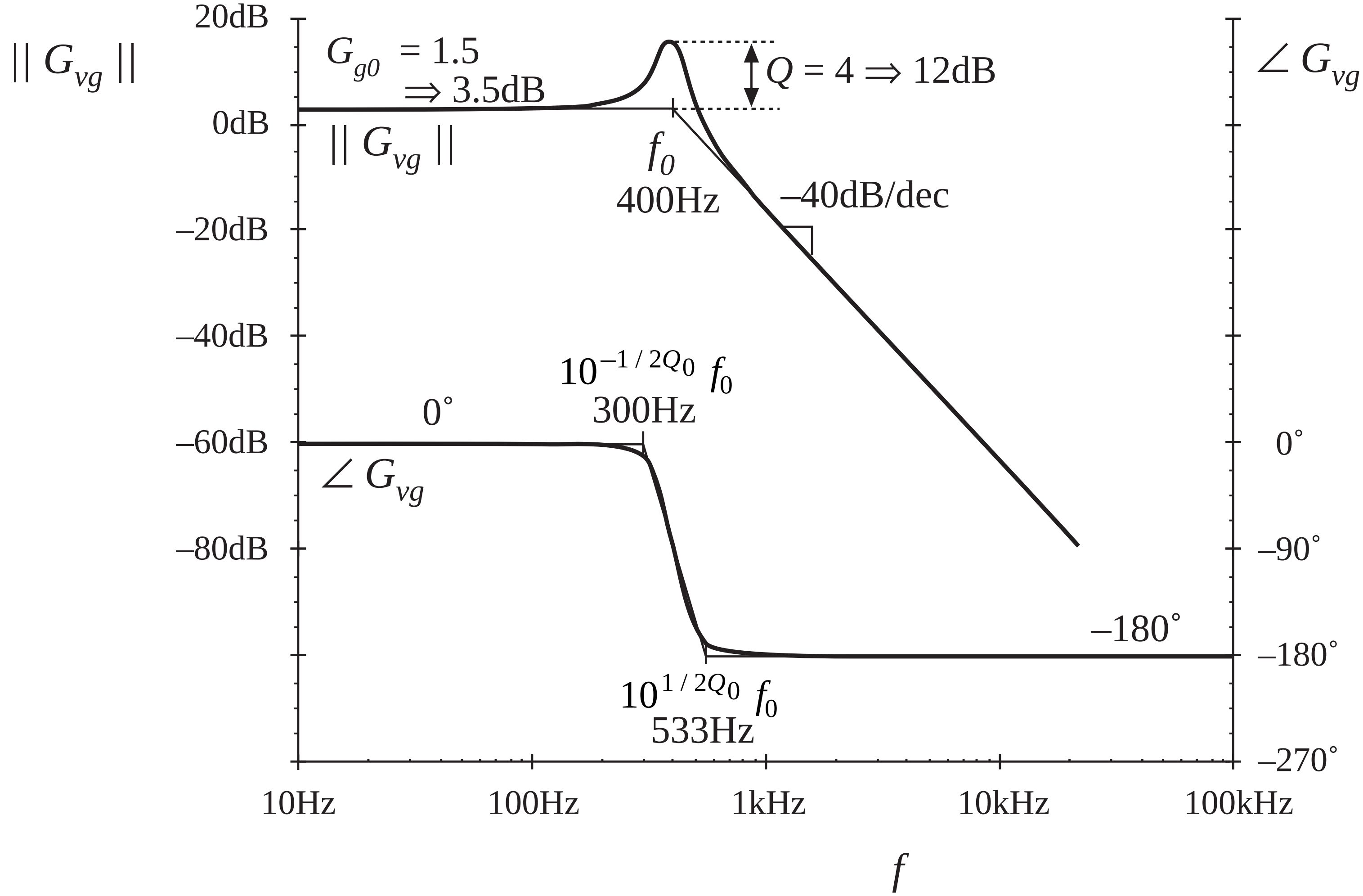
Then the salient features have the following numerical values:

$$
\begin{aligned}
\left| G_{g0} \right| &= \frac{D}{D'} = 1.5 \Rightarrow 3.5\text{dB} \\
\left| G_{d0} \right| &= \frac{\left| V \right|}{D\,D'^2} = 469\text{V} \Rightarrow 53.4\text{dBV} \\
f_0 &= \frac{\omega_0}{2\pi} = \frac{D'}{2\pi\sqrt{LC}} = 400\text{Hz} \\
Q &= D'R\sqrt{\frac{C}{L}} = 4 \Rightarrow 12\text{dB} \\
f_z &= \frac{\omega_z}{2\pi} = \frac{D'R}{2\pi DL} = 6.6\text{kHz}
\end{aligned}
$$

Bode plot: control-to-output transfer function



Bode plot: line-to-output transfer function



8.2.2. Transfer functions of some basic CCM converters

Table 8.2. Salient features of the small-signal CCM transfer functions of some basic dc-dc converters

Converter	G_{g0}	G_{d0}	ω_0	Q	ω_z
buck	D	$\frac{V}{D}$	$\frac{1}{\sqrt{LC}}$	$R\sqrt{\frac{C}{L}}$	∞
boost	$\frac{1}{D'}$	$\frac{V}{D'}$	$\frac{D'}{\sqrt{LC}}$	$D'R\sqrt{\frac{C}{L}}$	$\frac{D'^2R}{L}$
buck-boost	$-\frac{D}{D'}$	$\frac{V}{D\,D'^2}$	$\frac{D'}{\sqrt{LC}}$	$D'R\sqrt{\frac{C}{L}}$	$\frac{D'^2R}{D\,L}$

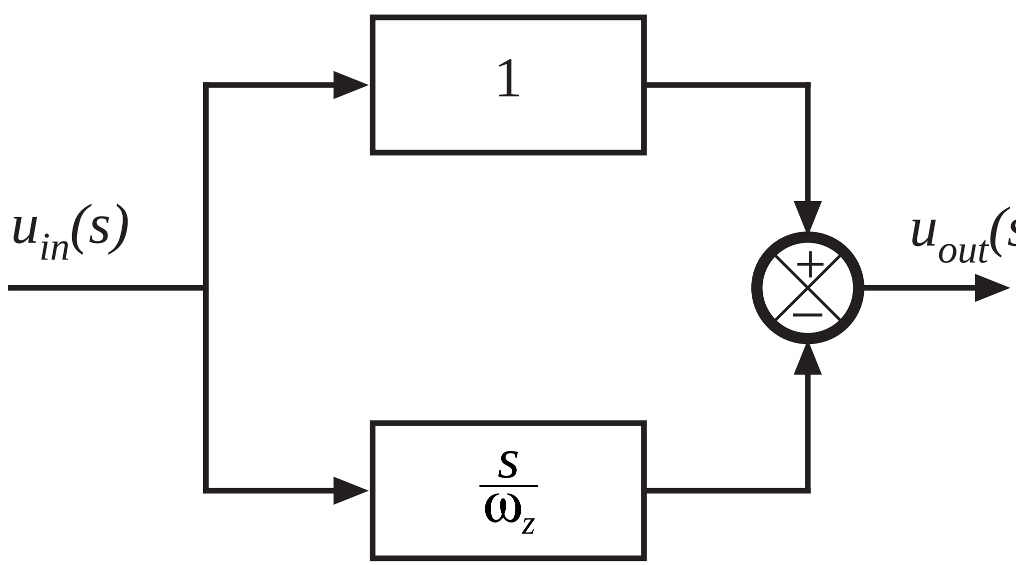
where the transfer functions are written in the standard forms

$$G_{vd}(s) = G_{d0}\,\frac{\left(1 - \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2\right)}$$

$$G_{vg}(s) = G_{g0}\,\frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

8.2.3. Physical origins of the right half-plane zero

$$G(s) = \left(1 - \frac{s}{\omega_0}\right)$$

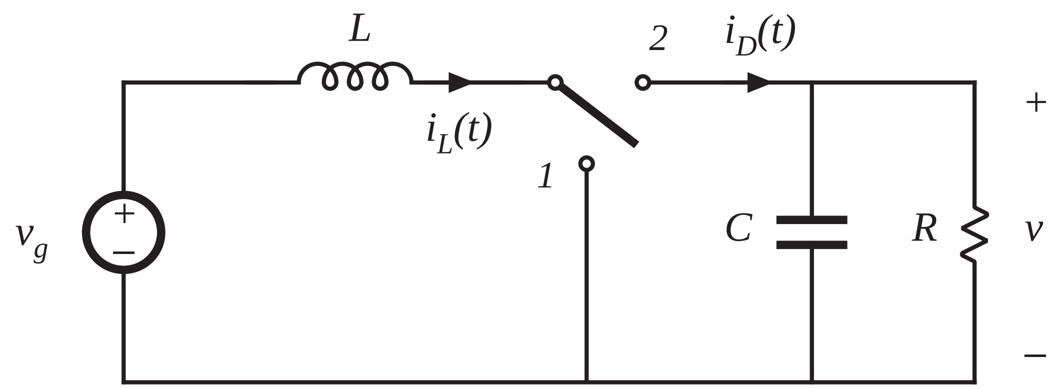


- *phase reversal at high frequency*
- *transient response: output initially tends in wrong direction*

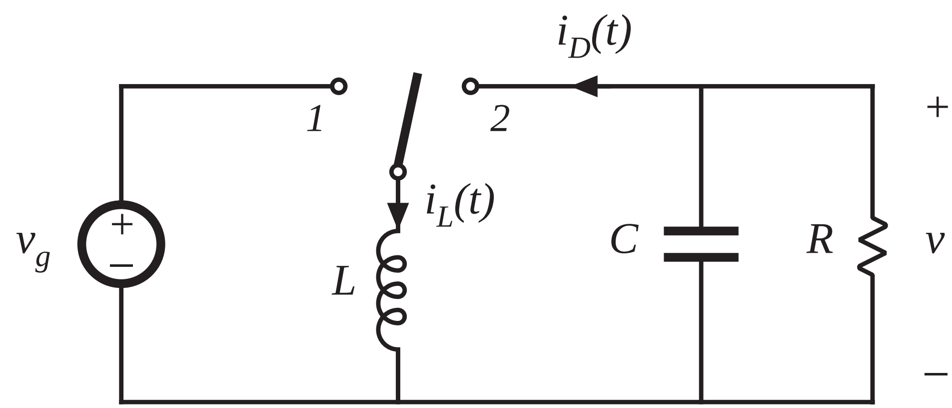
Two converters whose CCM control-to-output transfer functions exhibit RHP zeroes

$$\left\langle i_D \right\rangle_{T_s} = d' \left\langle i_L \right\rangle_{T_s}$$

Boost



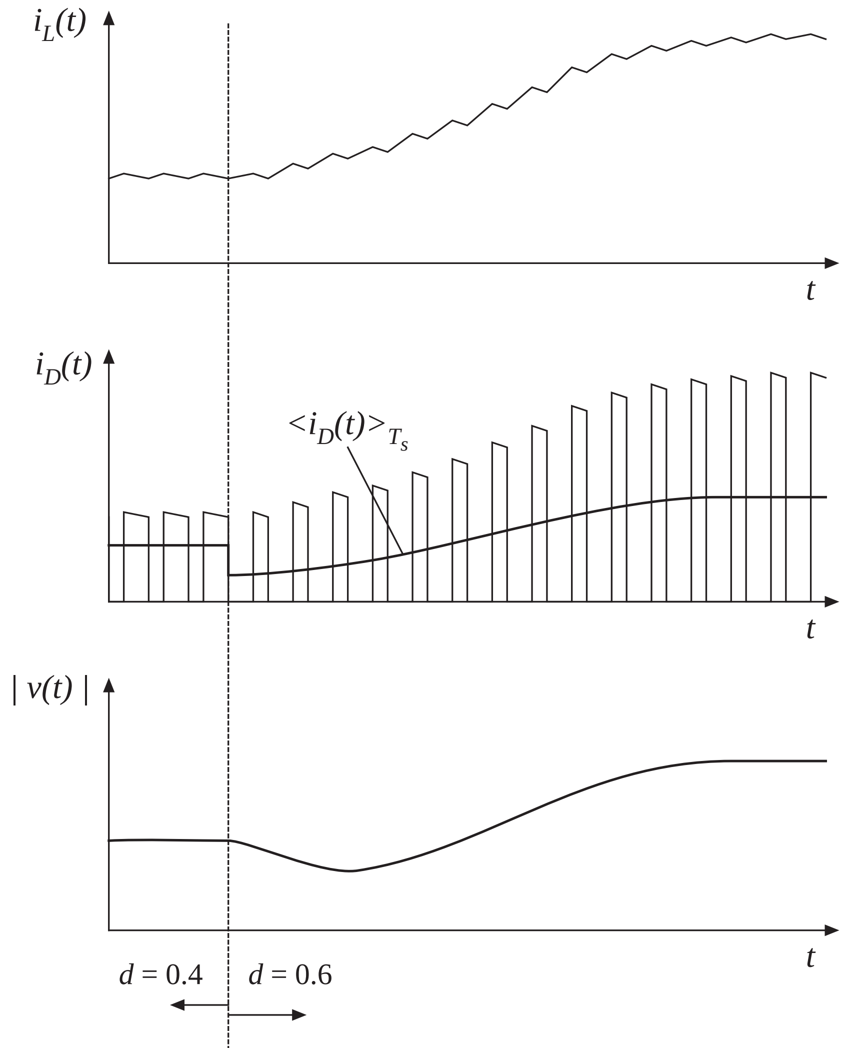
Buck-boost



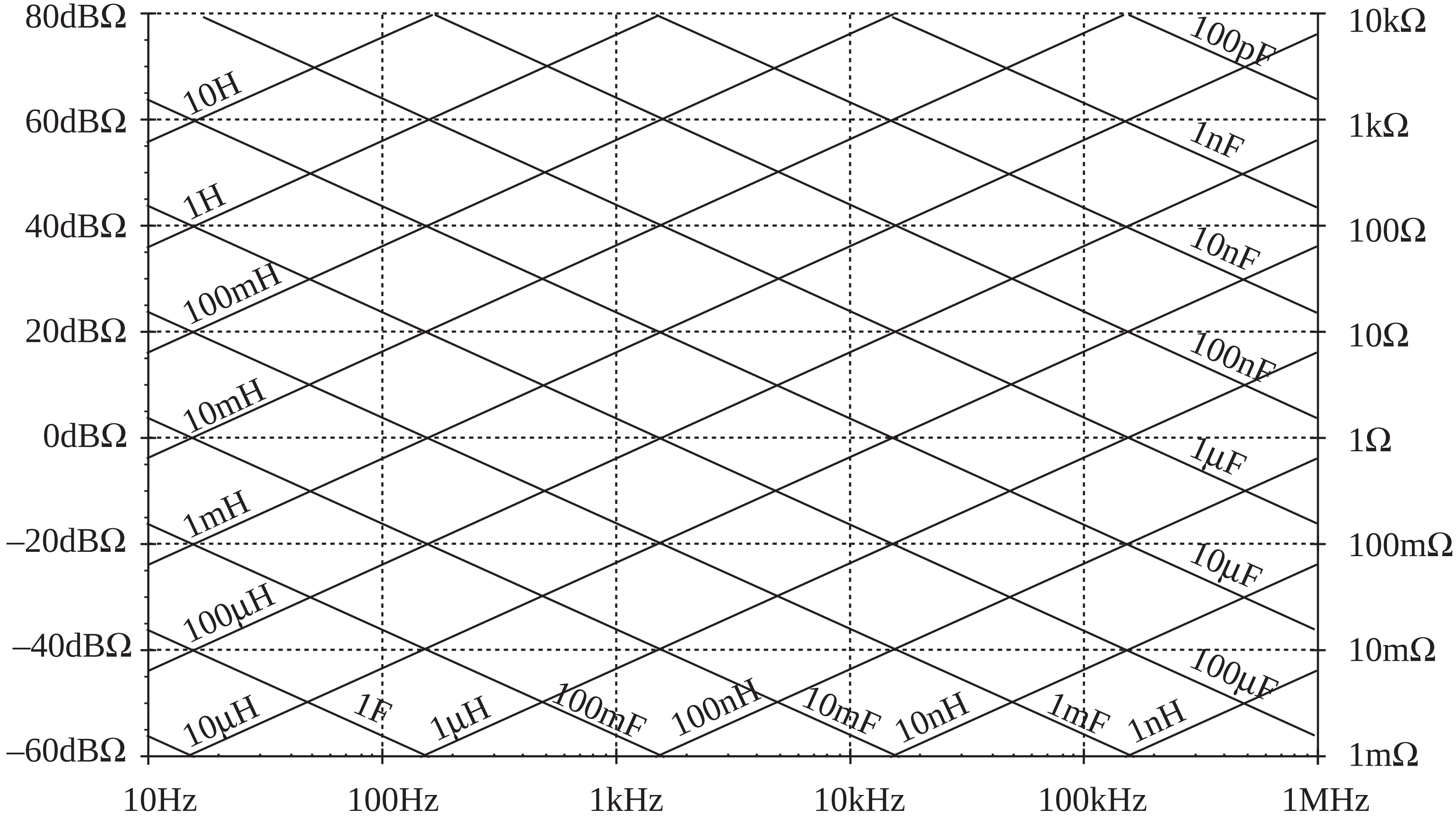
Waveforms, step increase in duty cycle

$$\langle i_D \rangle_{T_s} = d' \langle i_L \rangle_{T_s}$$

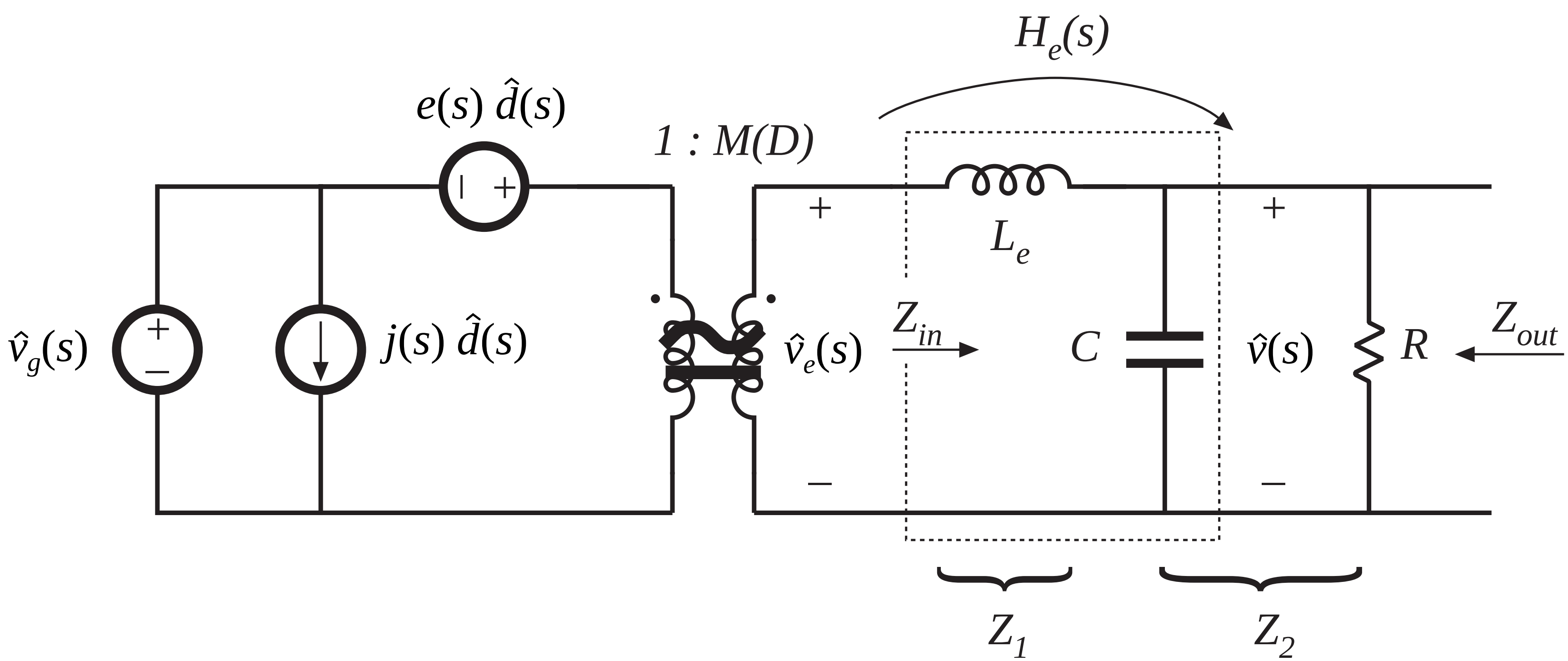
- Increasing *d(t)* causes the average diode current to initially decrease
- As inductor current increases to its new equilibrium value, average diode current eventually increases



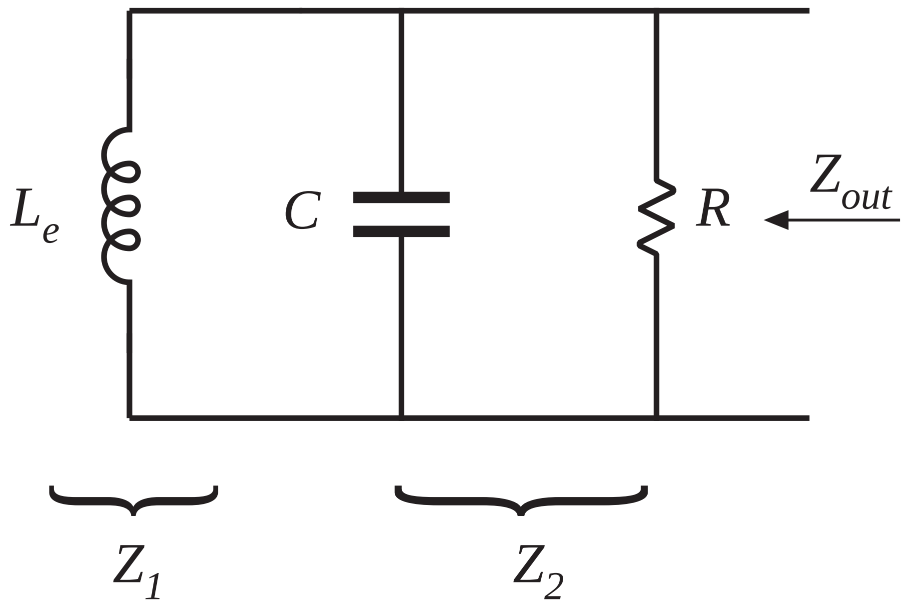
Impedance graph paper



Transfer functions predicted by canonical model

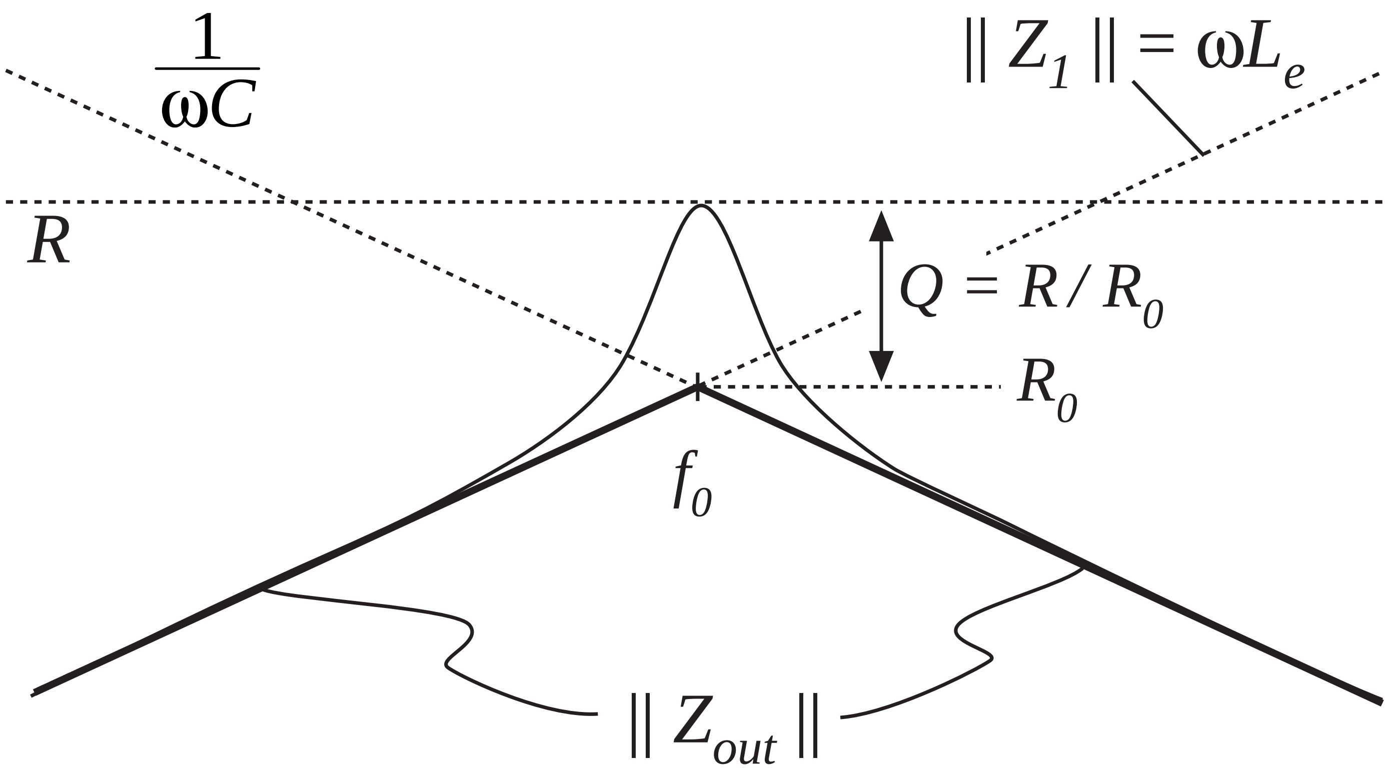


Output impedance Z_{out}: set sources to zero

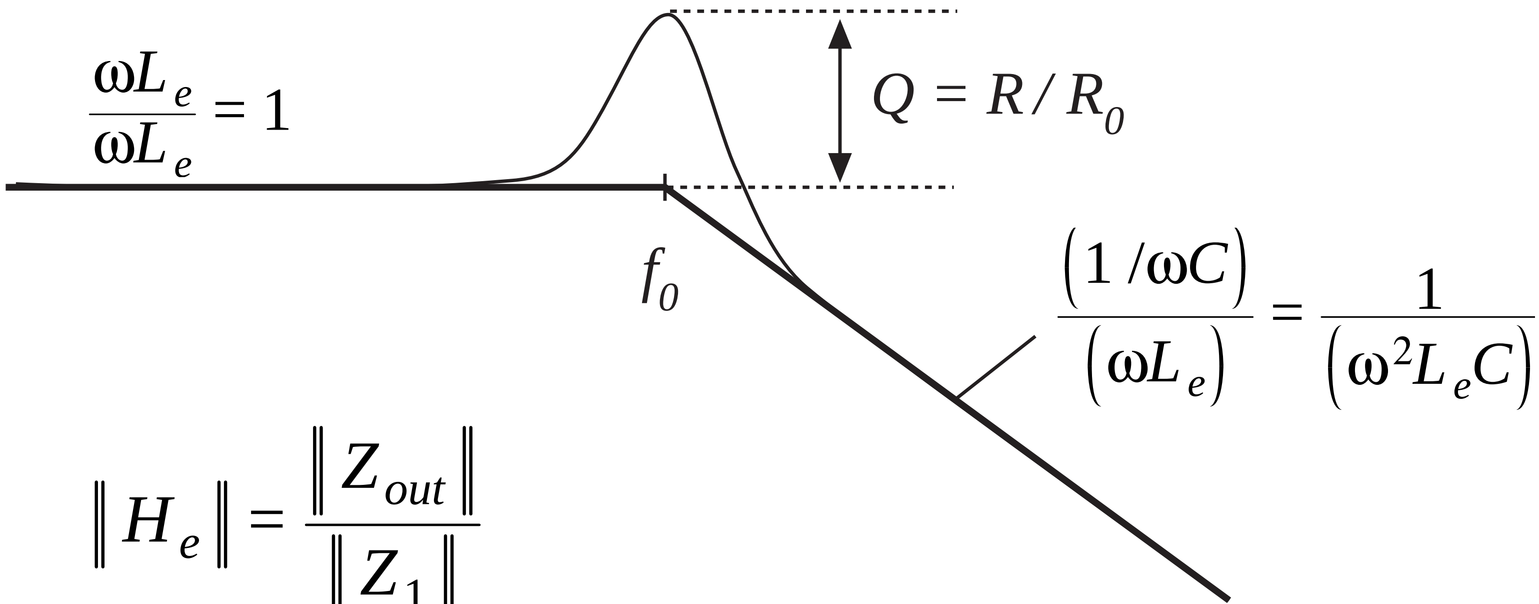


$$Z_{out} = Z_1 \parallel Z_2$$

Graphical construction of output impedance

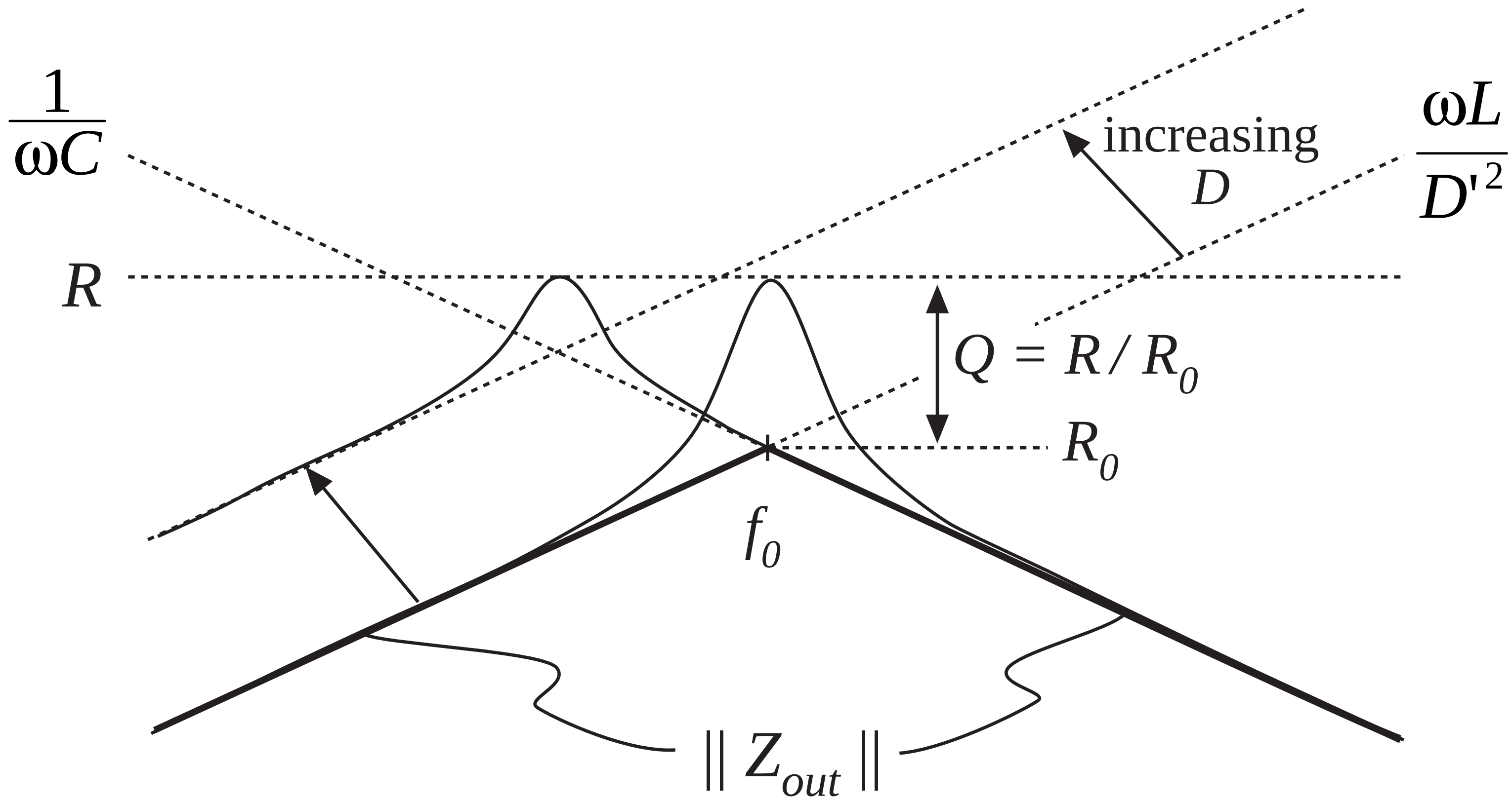


Graphical construction of filter effective transfer function

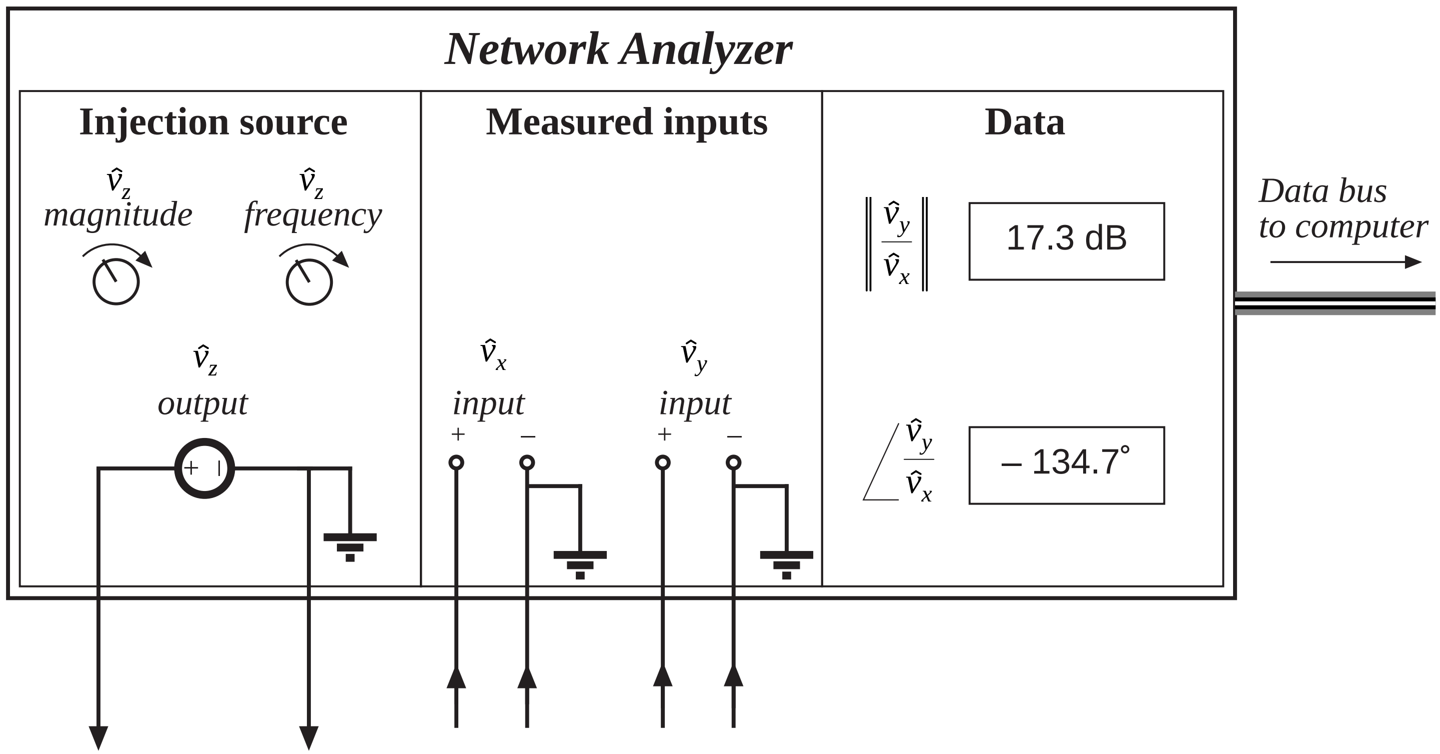


$$\| H_e \| = \frac{\| Z_{out} \|}{\| Z_1 \|}$$

Boost and buck-boost converters: $L_e = L \,/\, D'^{\,2}$



8.4. Measurement of ac transfer functions and impedances



Swept sinusoidal measurements

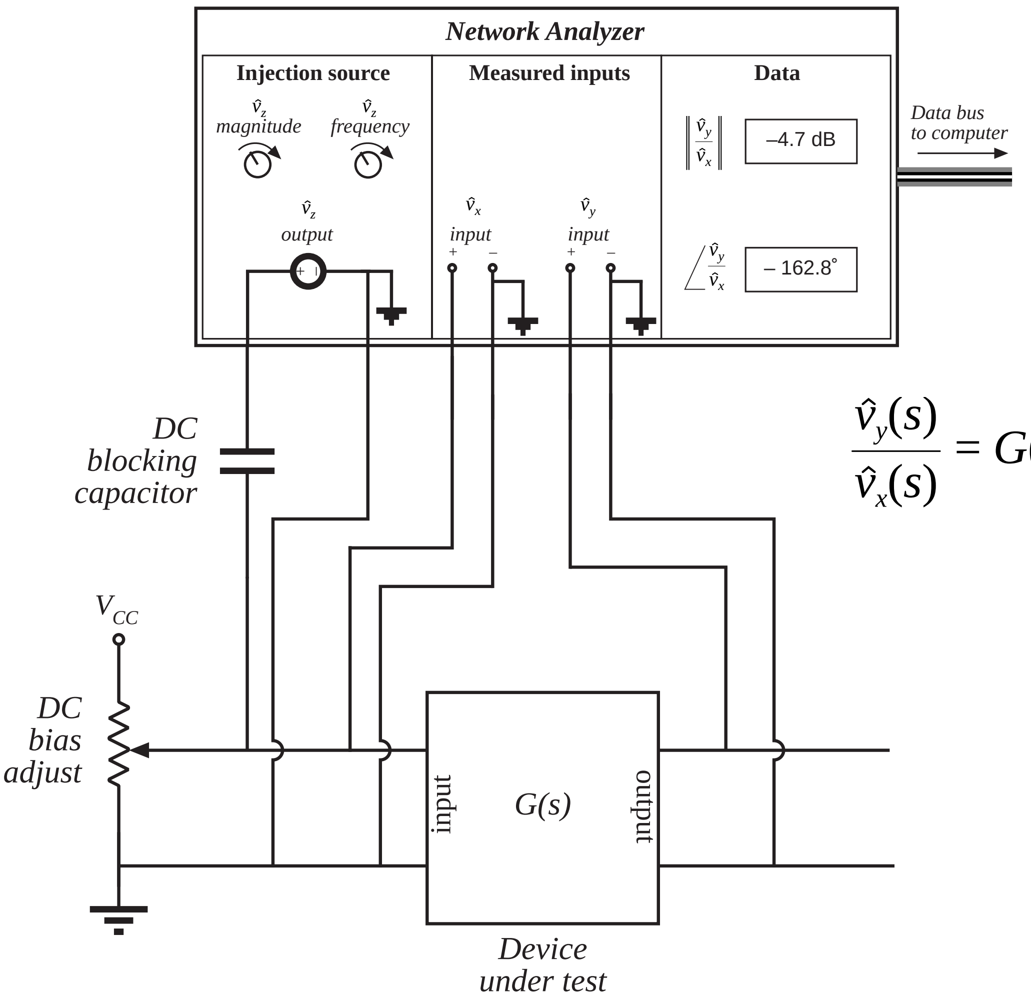
- Injection source produces sinusoid $\hat{v}_z$ of controllable amplitude and frequency
- Signal inputs $\hat{v}_x$ and $\hat{v}_y$ perform function of narrowband tracking voltmeter:

  Component of input at injection source frequency is measured

  Narrowband function is essential: switching harmonics and other noise components are removed
- Network analyzer measures

$$\left\| \frac{\hat{v}_y}{\hat{v}_x} \right\| \qquad \text{and} \qquad \angle \frac{\hat{v}_y}{\hat{v}_x}$$

Measurement of an ac transfer function

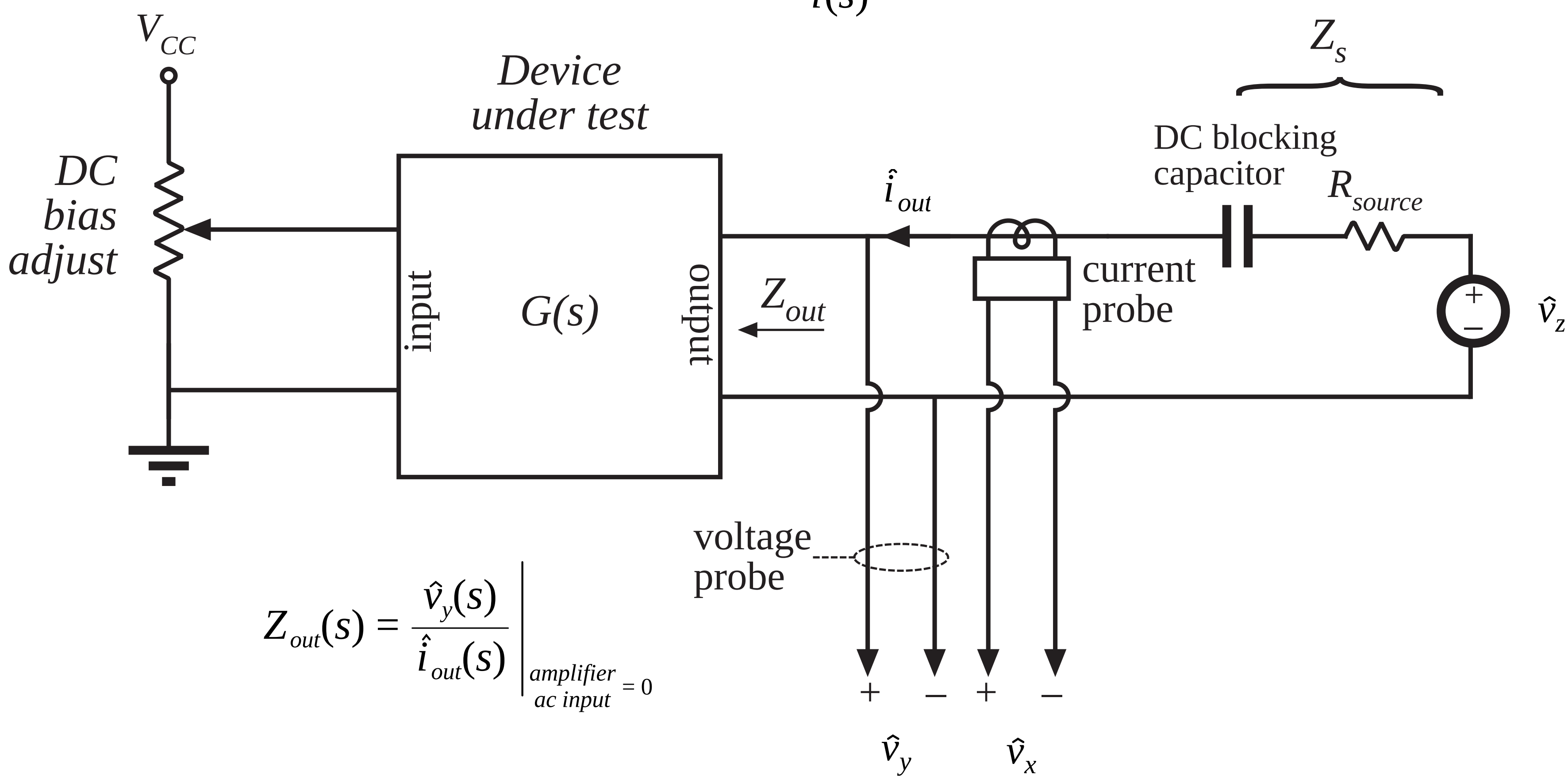


$$\frac{\hat{v}_y(s)}{\hat{v}_x(s)} = G(s)$$

- Potentiometer establishes correct quiescent operating point
- Injection sinusoid coupled to device input via dc blocking capacitor
- Actual device input and output voltages are measured as $\hat{v}_x$ and $\hat{v}_y$
- Dynamics of blocking capacitor are irrelevant

Measurement of an output impedance

$$Z(s) = \frac{\hat{v}(s)}{\hat{i}(s)}$$



$$Z_{out}(s) = \left. \frac{\hat{v}_y(s)}{\hat{i}_{out}(s)} \right|_{\substack{\text{amplifier} \\ \text{ac input}} = 0}$$

Measurement of output impedance

- Treat output impedance as transfer function from output current to output voltage:

$$Z(s) = \frac{\hat{v}(s)}{\hat{i}(s)} \qquad Z_{out}(s) = \left. \frac{\hat{v}_y(s)}{\hat{i}_{out}(s)} \right|_{\substack{amplifier \\ ac\ input}\, = 0}$$

- Potentiometer at device input port establishes correct quiescent operating point
- Current probe produces voltage proportional to current; this voltage is connected to network analyzer channel $\hat{v}_x$
- Network analyzer result must be multiplied by appropriate factor, to account for scale factors of current and voltage probes

Measurement of small impedances

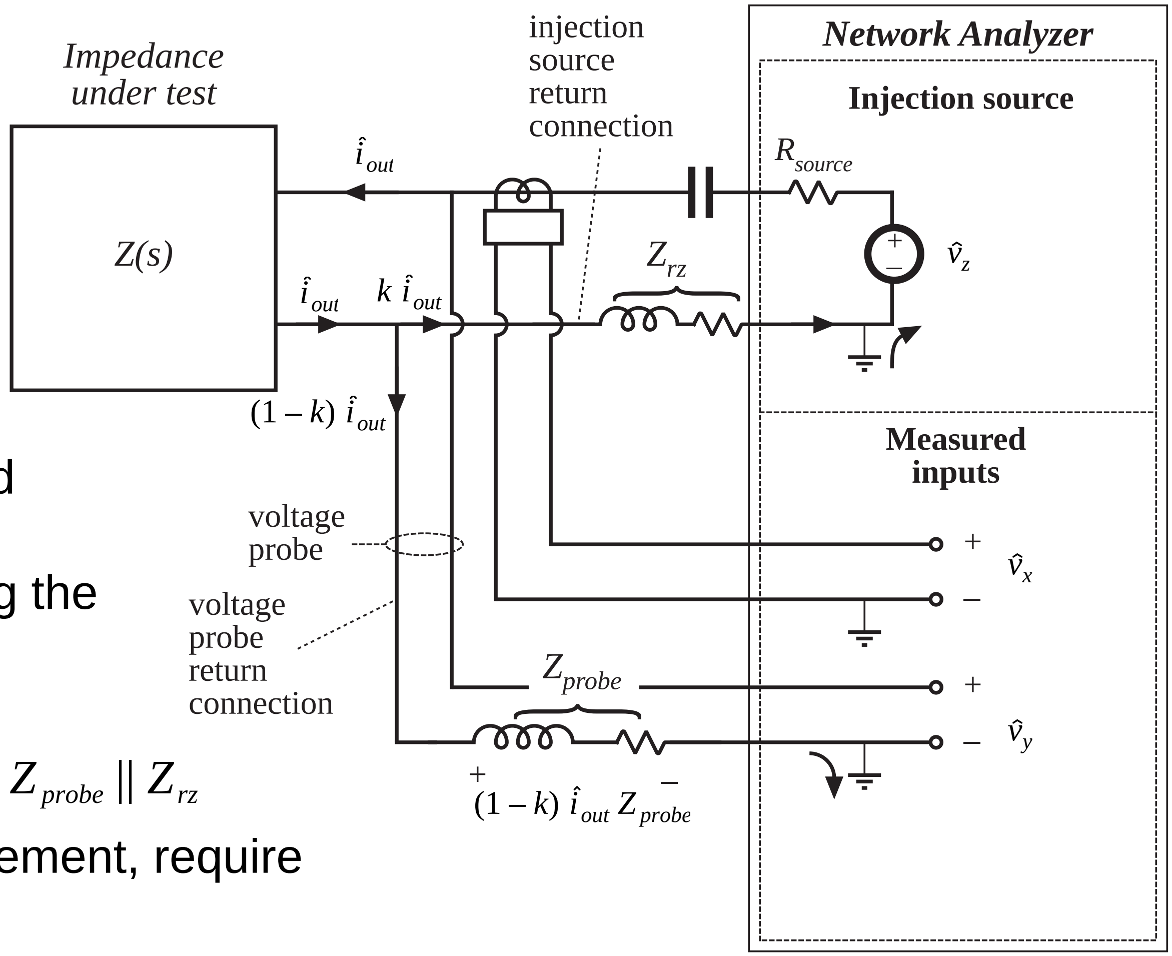
Grounding problems cause measurement to fail:

Injection current can return to analyzer via two paths. Injection current which returns via voltage probe ground induces voltage drop in voltage probe, corrupting the measurement. Network analyzer measures

$$Z + (1 - k)\, Z_{probe} = Z + Z_{probe} \parallel Z_{rz}$$

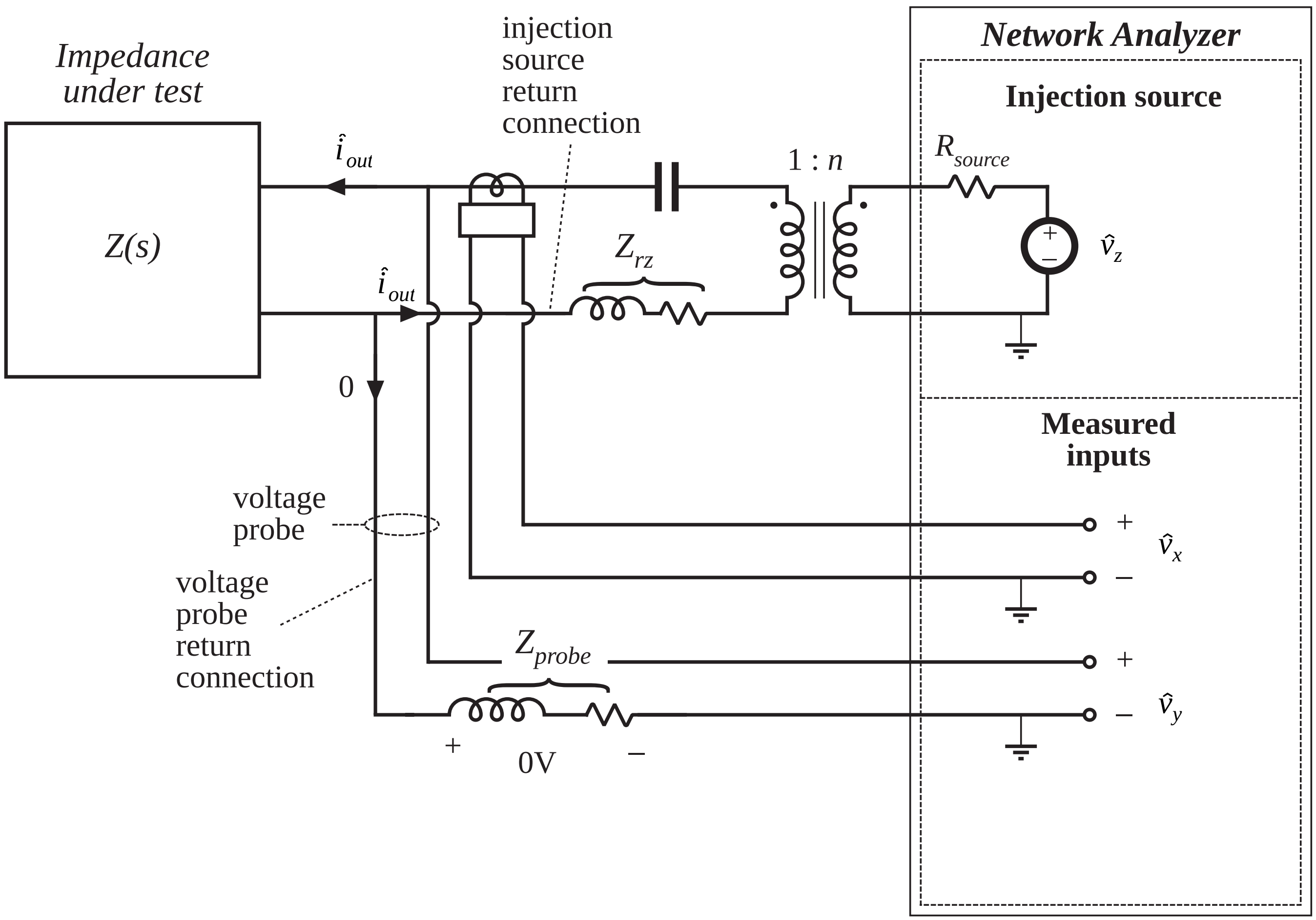
For an accurate measurement, require

$$\left\| Z \right\| >> \left\| \left(Z_{probe} \parallel Z_{rz} \right) \right\|$$



Improved measurement: add isolation transformer

Injection current must now return entirely through transformer. No additional voltage is induced in voltage probe ground connection



8.5. Summary of key points

1. The magnitude Bode diagrams of functions which vary as $(f / f_0)n$ have slopes equal to $20n$ dB per decade, and pass through 0dB at $f = f_0$.
2. It is good practice to express transfer functions in normalized pole-zero form; this form directly exposes expressions for the salient features of the response, i.e., the corner frequencies, reference gain, etc.
3. The right half-plane zero exhibits the magnitude response of the left half-plane zero, but the phase response of the pole.
4. Poles and zeroes can be expressed in frequency-inverted form, when it is desirable to refer the gain to a high-frequency asymptote.

Summary of key points

5. A two-pole response can be written in the standard normalized form of Eq. (8-53). When $Q > 0.5$, the poles are complex conjugates. The magnitude response then exhibits peaking in the vicinity of the corner frequency, with an exact value of Q at $f = f_0$. High Q also causes the phase to change sharply near the corner frequency.

6. When the Q is less than 0.5, the two pole response can be plotted as two real poles. The low- Q approximation predicts that the two poles occur at frequencies f_0 / Q and Qf_0. These frequencies are within 10% of the exact values for $Q \leq 0.3$.

7. The low- Q approximation can be extended to find approximate roots of an arbitrary degree polynomial. Approximate analytical expressions for the salient features can be derived. Numerical values are used to justify the approximations.

Summary of key points

8. Salient features of the transfer functions of the buck, boost, and buck-boost converters are tabulated in section 8.2.2. The line-to-output transfer functions of these converters contain two poles. Their control-to-output transfer functions contain two poles, and may additionally contain a right half-pland zero.

9. Approximate magnitude asymptotes of impedances and transfer functions can be easily derived by graphical construction. This approach is a useful supplement to conventional analysis, because it yields physical insight into the circuit behavior, and because it exposes suitable approximations. Several examples, including the impedances of basic series and parallel resonant circuits and the transfer function $H_e(s)$ of the boost and buck-boost converters, are worked in section 8.3.

10. Measurement of transfer functions and impedances using a network analyzer is discussed in section 8.4. Careful attention to ground connections is important when measuring small impedances.