Choosing a Frequency Multiplier's Waveform
by
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Any non-sinewave repetitive waveform contains energy at harmonics of the fundamental frequency. The multiplier designer's task is to create a non-linear circuit that produces a waveform with significant signal strength at the desired harmonics. "Seat-of-the-pants" experiments at the bench with a spectrum analyzer or computer simulator can produce excellent results but a little insight into the mathematics underlying harmonic generation will guide the designer to a quicker solution. Perhaps the most powerful mathematical tool for evaluating candidate waveforms is the Fourier Series approximation. Fourier analysis is a straightforward method for determining the harmonic content of a repetitive waveform since the Fourier Series approximates a waveform as a summation of harmonically related sinewaves :

$$F(\theta) = A_0 + A_1 \sin(\theta) + ... + A_N \sin(n\theta) + B_1 \cos(\theta) + ... + B_N \cos(n\theta) \qquad (1\text{-}1)$$

For multiplier purposes, the A_0 term may be ignored since it represents the average "DC" value of the waveform. Also, each harmonic has a sine and cosine component which may be combined to form a single sine or cosine term with a new amplitude and a phase shift. Only the amplitude is of interest when frequency multiplying :

$$F(\theta)_{ac} = C_1 \cos(\theta) + ... + C_N \cos(n\theta) \qquad (1\text{-}2)$$

Although this equation is incomplete since the DC term and the phase shift terms in parentheses were left out, the multiplier designer is only concerned with the amplitudes, C_n, which are calculated as the square root of the sum of the squares of A_n and B_n. Many repetitive waveforms are symmetrical about the y-axis so by proper selection of the $\theta = 0$ point either the A_n or B_n terms can be made equal to zero and the remaining terms become the C_n terms. The A_n terms are calculated by integrating the candidate waveform multiplied by an nth harmonic sinewave over one cycle of the waveform and the B_n terms are the integral of the waveform multiplied by an nth harmonic cosine:

$$A_n = 1/\pi \int F(\theta) \sin(n\theta) \, d\theta \quad (\text{ integrated over one cycle }) \qquad (1\text{-}3)$$

$$B_n = 1/\pi \int F(\theta) \cos(n\theta) \, d\theta \quad (\text{ integrated over one cycle }) \qquad (1\text{-}4)$$
where $F(\theta)$ is the waveform.

These integrals are often easy to calculate and some may be done by inspection. For example, consider a very narrow pulse ($F(\theta)$ = pulse). Such a short pulse has the familiar comb-like spectrum with equal amplitude harmonics. Note that the A_n terms will be near zero since the sine function doesn't have time to rise before the short pulse is over ($\sin(0) = 0$). The cosine function starts at one ($\cos(0) = 1$) and doesn't have time to drop before the pulse ends. Thus all of the cosine integrals have about the same value - hence the flat spectrum.

Fig. 1 shows the amplitude terms (peak value of the nth harmonic sinewave) for various waveforms calculated using the above equations. It is obvious that waveforms with fast edges have larger high frequency harmonics: notice that the harmonics without vertical edges have n^2 in the denominator but the waveforms with fast edges only have n in the denominator. It can be seen that the timing between the positive and negative edges of a pulse determines which harmonics are emphasized. For example, a 50/50 squarewave has only odd harmonics: the timing is wrong for the buildup of even-harmonic energy but a 25/75 duty-cycle contains large even harmonics: the edges occur at the right time to reinforce certain even harmonics.
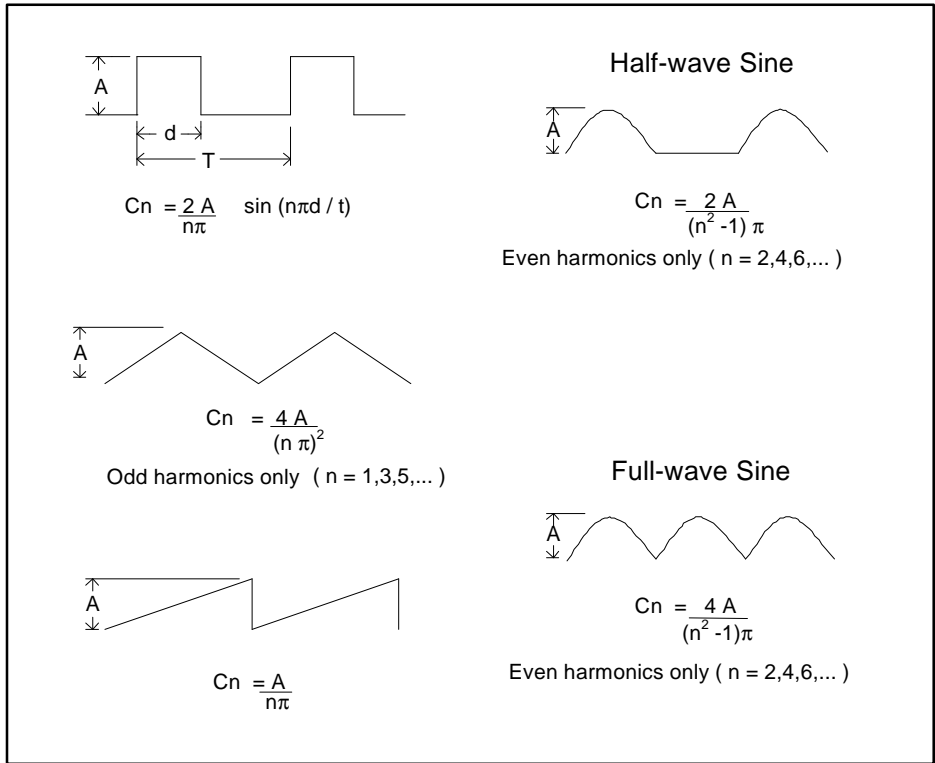
Figure 1 (waveforms):

Pulse train (amplitude A, width d, period T):
$$C_n = \frac{2A}{n\pi}\sin(n\pi d/t)$$

Triangle wave (amplitude A):
$$C_n = \frac{4A}{(n\pi)^2}$$
Odd harmonics only (n = 1,3,5,...)

Sawtooth (amplitude A):
$$C_n = \frac{A}{n\pi}$$

Half-wave Sine (amplitude A):
$$C_n = \frac{2A}{(n^2-1)\pi}$$
Even harmonics only (n = 2,4,6,...)

Full-wave Sine (amplitude A):
$$C_n = \frac{4A}{(n^2-1)\pi}$$
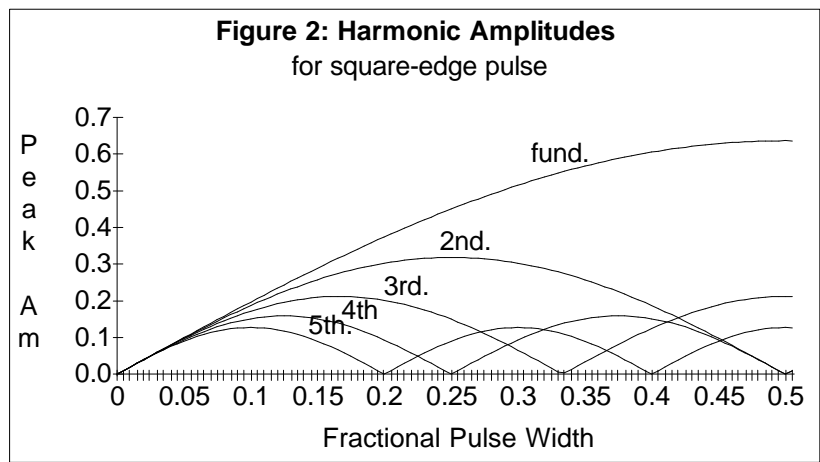Even harmonics only (n = 2,4,6,...)

Figure 1: Harmonic amplitude terms for various waveforms.

Fig. 2 shows the harmonic content of a square pulse as a function of its duty-cycle. To illustrate the chart's utility consider the design of a frequency doubler using a square-sided pulse. The chart suggests that the most second harmonic energy will be generated when the duty-cycle is 25/75. But it can also be seen that if the duty cycle is increased to 33/67 then the third harmonic drops to zero which could simplify output filtering with little drop in the desired second harmonic.

Figure 2: Harmonic Amplitudes
for square-edge pulse

Peak Am [vertical axis: Peak Amplitude, 0.0 to 0.7]

Curves labeled: fund., 2nd., 3rd., 4th, 5th.

Fractional Pulse Width [horizontal axis: 0 to 0.5]

Figs. 3 and 4 show the first twelve harmonics for selected duty-cycles illustrating how certain harmonics may be nullified by proper selection of the duty cycle. Deep nulls are posssible but the duty-cycle must remain stable over time and temperature. Also consider that the phase noise performance can be seriously degraded by many pulse-width generating circuits. It is most desireable for the input signal to control the timing of both edges of the squarewave instead of one edge being controlled by an R-C timing network as with monostable

multivibrators. Reactive phase shift networks or coaxial delay lines may be used to generate the desired time delay with good results.Rectified sinewaves contain a large second harmonic with quickly dropping higher harmonics so very efficient diode doublers are straightforward. Full-wave versions are usually preferred since the fundamental frequency is greatly reduced, approaching zero in well balanced circuits.

Figure 3: Pulse Harmonic

50/50 duty cycle square-wave

Figure 4: Pulse Harmonic

25% duty cycle
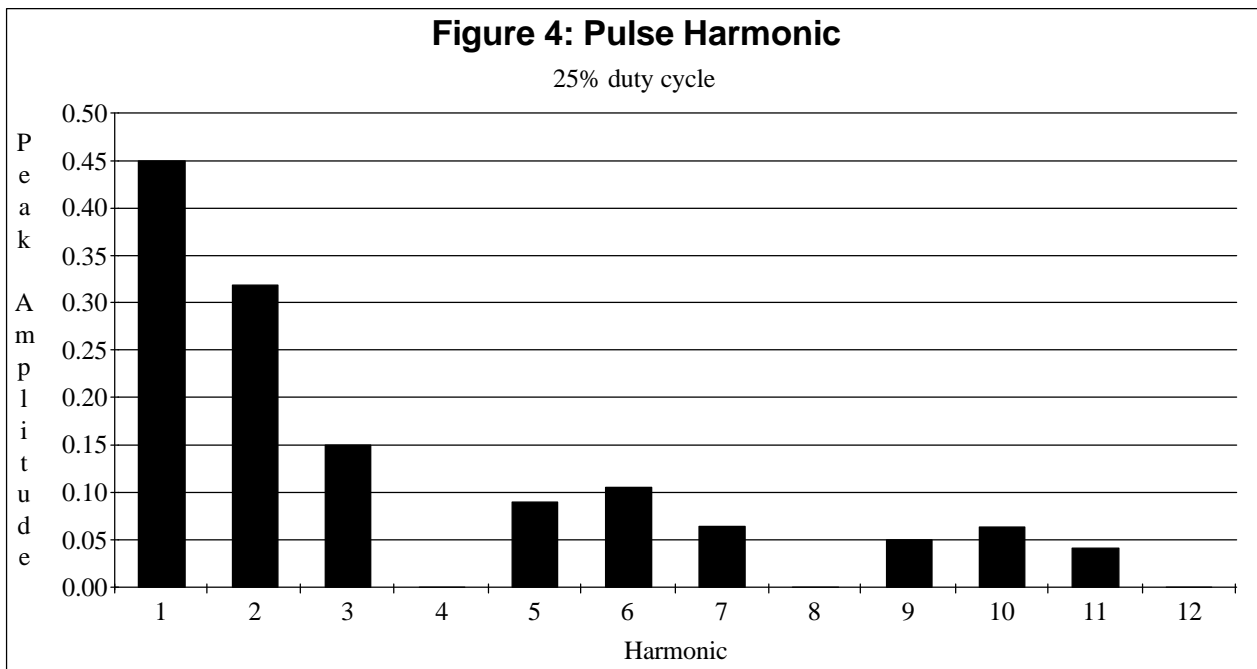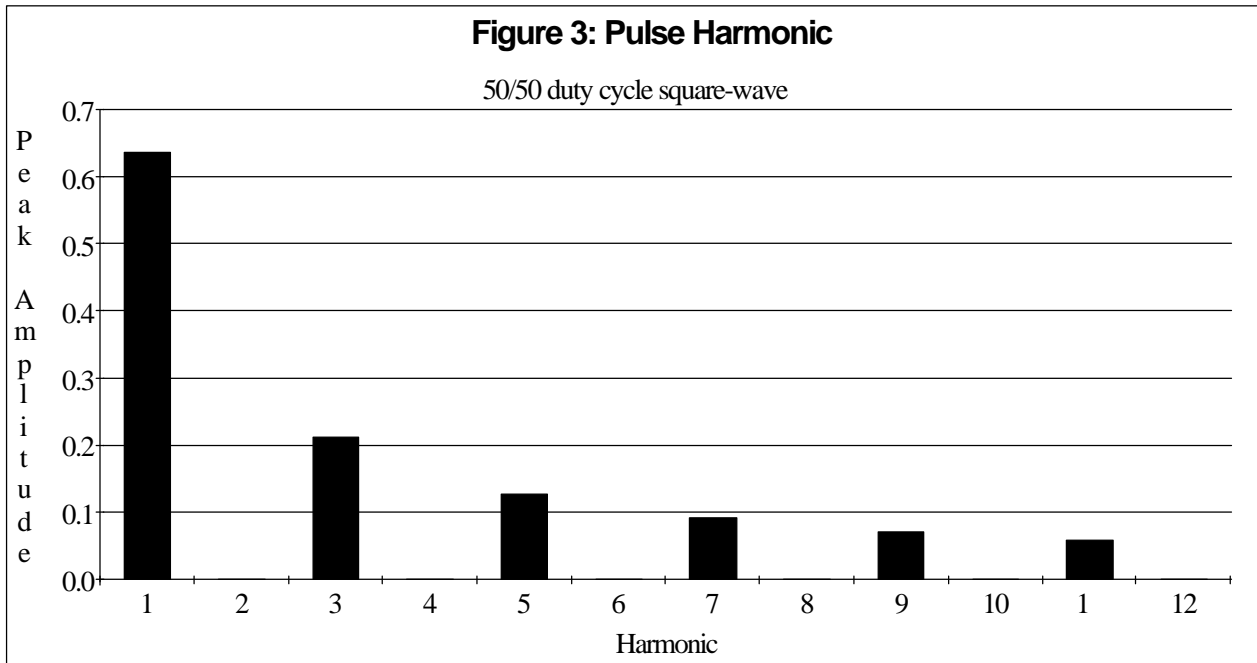
In most cases the Fourier coefficients will accurately predict the output signal size but simplistic waveform analysis can be misleading when the harmonic selection network interacts with the generator. For example, the pulse generated by a step recovery diode with a resistive load has energy spread across a large spectrum with little energy concentrated at any particular harmonic but a properly designed multiplier will reflect the energy of the undesired harmonics back to the diode for "reprocessing" resulting in efficient conversion of the input energy to the desired output frequency. In this example the reflected harmonics "change" the input waveform in a beneficial way but usually the design challenge is to prevent the harmonic selection network from loading the waveform source in an undesireable or excessive manner while providing sufficient filtering of undesired harmonics. In general terms, the harmonic selection circuit should present a low impedance at the deisred harmonic and a high impedance at the undesired harmonics to low impedance waveform sources (voltage-like sources). When the source is a high impedance or current-like as with some odd-order diode multipliers or transistor collectors the harmonic selection network should exhibit a low impedance at the undesired harmonics and a high impedance at the desired frequency. There are, no doubt, exceptions to these rules but quite often multiplier circuits will have mysterious "idler" tanks and traps and seemingly unnecessary tuned circuits, capacitors and inductors included in the design to essentially comply or approximate compliance with these rules enhancing the desired harmonic production.