

**THE ELLIPTIC FILTER IS THE BEST CHOICE FOR MEETING STRINGENT MAGNITUDE-RESPONSE REQUIREMENTS, AND A DESIGN TECHNIQUE HELPS YOU TO MAXIMIZE THE BAND-EDGE SELECTIVITY WITHOUT INCREASING FILTER ORDER.**

# Designing elliptic filters with maximum selectivity

**E**LECTRONIC-FILTER DESIGN, whether analog, digital, or distributed, is an essential part of many electrical engineers' workdays. Frequency-selective networks are useful for suppressing noise, rejecting unwanted signals, or in some way manipulating the input signal's characteristics. Although applications abound, engineers typically use classical filters that are polynomial approximations to the brick-wall filter (see sidebar "A new look at the brick-wall filter"). These classical filters include Butterworth, Chebyshev, and elliptic filters.

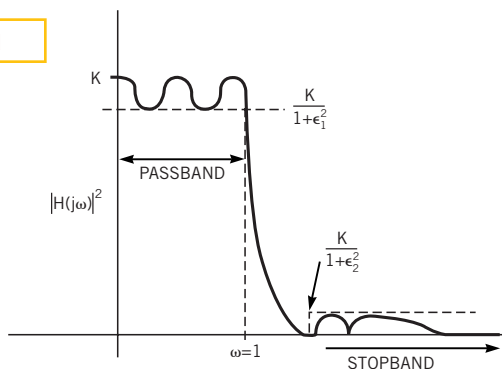
Filter requirements often call for highly selective filters, especially in bandpass filters designed to reject out-of-band carriers. If the cutoff-rate specification is stringent, the classical Butterworth and Chebyshev filters result in high orders. A higher order adds complexity to the filter, and the resulting design is more difficult to tune. The sensitivity of the filter to its components also increases. These issues apply to both lumped-element realizations and microwave structures. For microwave structures, the physical features of the implementation directly influence the overall characteristics of the filter.

When selectivity is an issue, you can rely on elliptic filters, which provide the lowest order implementation of the classical filters for the same frequency and rejection requirements. Elliptic filters are equiripple in the passband and the stopband (Figure 1). The finite zeros of transmission, which allow the filter to have a narrower transition band, determine the ripple response in the stopband. The price of a narrower transition band is asymptotic roll-offs of  $-20$  (order  $n$  odd) or  $-40$  (order  $n$  even)

dB/decade (Reference 1) and the additional complexity of achieving the transmission zeros.

Despite these limitations, the elliptic filter is the filter of choice for stringent magnitude-response requirements. The elliptic filter has the additional advantage of providing several degrees of freedom for controlling its response, including band-edge selectivity. Many designers resort to ad hoc and often wasteful techniques to obtain superior selectivity. However, a new technique allows you to maximize the band-edge selectivity (BES) of elliptic filters without increasing filter order. The technique effectively narrows the transition band by moving the notch frequency closer to the passband. This change increases the lobe levels to the original stopband-rejection requirement and impacts delay perform-

**Figure 1**



**The magnitude response of an elliptic filter shows equal ripple in the passband and stopband.**

ance in the passband. A design example shows the ease with which you can design elliptic filters with maximum selectivity without increasing filter order. By maximizing the selectivity without increasing the filter order, you can reject more noise or unwanted signal components closer to the band edge—a desirable function.

**MAKE BETTER FILTERS WITH NO ADDED COST**

You can use a recently derived formulation for the band-edge selectivity of elliptic filters and use a method for maximizing selectivity without increasing the filter order (Reference 2). This useful method, in conjunction with the sensitivity calculations, can result in superior filters at no additional cost. The following design example highlights the power and ease of this method.

The BES of a filter is:

$$\text{BES} = - \left. \frac{d|H(j\omega)|}{d\omega} \right|_{\omega=1} \quad (1)$$

The selectivity is the slope of the magnitude response of the filter at the normalized corner frequency, or band edge. Selectivity is a measure of the cutoff rate, and the “larger-the-better” characteristic applies here. Most designers generally accept selectivity as a property of a filter and not as a goal of filter design. However, you can treat filter selectivity as a design parameter that you can optimize.

The BES of an elliptic filter is (Reference 2)

$$\text{BES} = \left( \frac{\epsilon_1^2 n^2}{(1 + \epsilon_1^2)^{3/2}} \right) \left( \frac{1 - m'}{1 - m} \right), \quad (2)$$

where  $n$  is the order,  $\epsilon_1$  is the passband ripple parameter, and

$$m = \frac{1}{\Omega_s^2}, \quad m' = \left( \frac{\epsilon_1}{\epsilon_2} \right)^2 \quad (3)$$

$\Omega_s$  is the stopband corner frequency, and  $\epsilon_2$  is the stopband ripple parameter (Figure 1). If you’re familiar with filter theory, you’ll recognize the first term in the parentheses of Equation 2 as the BES of the Chebyshev filter. However, for the elliptic filter, the new term  $(1 - m')/(1 - m)$  scales this selectivity. As  $m' \rightarrow 0$ , Equation 2 reduces to

$$\text{BES} = \left( \frac{\epsilon_1^2 n^2}{(1 + \epsilon_1^2)^{3/2}} \right) \left( \frac{\Omega_s^2}{\Omega_s^2 - 1} \right) \quad (4)$$

The result of Equation 4 is that the BES of an elliptic filter is greater than that of the Chebyshev filter for any  $\Omega_s > 1$ , given the same order and passband ripple. Figure 2 shows a plot of the scaling factor. If the passband and stopband ripple are fixed, then  $\Omega_s$  is the only degree of freedom for maximizing the BES without increasing the filter order  $n$ .

**REVIEW FILTER SENSITIVITIES**

Before describing the filter-maximization process, it is useful to review the sensitivities of the BES of the elliptic filter to the various filter parameters. Recall that, when a dependent variable,  $y$ , is a function of two or more independent variables,  $x_i$ , where  $i = 1, 2, \dots, N$ , the sensitivity of  $y$  with respect to  $x_i$  is as follows (Reference 3):

$$S_{x_i}^y = \frac{\partial y/y}{\partial x_i/x_i}, \quad i = 1, 2, \dots, N \quad (5)$$

You therefore need to calculate the partial derivatives of the BES with respect to the various filter parameters as follows:

$$\frac{\partial \text{BES}}{\partial n} = \frac{2\epsilon_1^2 n}{(1 + \epsilon_1^2)^{3/2}} \left( \frac{1 - m'}{1 - m} \right); \quad (6)$$

$$\frac{\partial \text{BES}}{\partial \epsilon_1} = \frac{\epsilon_1 n^2 (\epsilon_1^4 + (4 + \epsilon_2^2)\epsilon_1^2 - 2\epsilon_2^2)}{(1 + \epsilon_1^2)^{5/2} \epsilon_2^2 (1 - m)}; \quad (7)$$

$$\frac{\partial \text{BES}}{\partial \epsilon_2} = \frac{2\epsilon_1 n^2 (m')^{3/2}}{(1 + \epsilon_1^2)^{3/2} (1 - m)}; \text{ and} \quad (8)$$

$$\frac{\partial \text{BES}}{\partial \Omega_s} = -2 \frac{\epsilon_1^2 n^2 (1 - m')}{(1 + \epsilon_1^2) \Omega_s^3 (1 - m)^2} \quad (9)$$

These equations are fairly complicated. However, by calculating the sensitivity using Equation 5 you get simplified results (Reference 2):

$$S_n^{\text{BES}} = 2; \quad (10)$$

$$S_{\epsilon_1}^{\text{BES}} = \frac{\epsilon_1^4 + (4 + \epsilon_2^2)\epsilon_1^2 - 2\epsilon_2^2}{(1 + \epsilon_1^2)(\epsilon_1^2 - \epsilon_2^2)}; \quad (11)$$

$$S_{\epsilon_2}^{\text{BES}} = \frac{2m'}{1 - m'}; \text{ and} \quad (12)$$

$$S_{\Omega_s}^{\text{BES}} = -\frac{2}{\Omega_s^2 - 1} \quad (13)$$

In most applications, the filter order is fixed, and Equation 10 always holds. On the other hand, you can control the sensitivity of the BES with respect to the passband ripple parameter  $\epsilon_1$  using either  $\epsilon_1$  or  $\epsilon_2$ . By setting  $\lambda = \epsilon_1^2$ , the numerator of Equation 11 becomes a quadratic of the form  $\lambda^2 + (4 + \epsilon_2^2)\lambda - 2\epsilon_2^2 = 0$ . Solving for  $\lambda$ , you obtain

$$\epsilon_1^2 = \left( \frac{1}{2} \right) \left( -(4 + \epsilon_2^2) + \sqrt{(4 + \epsilon_2^2)^2 + 8\epsilon_2^2} \right) \quad (14)$$

Equation 14 strictly depends on  $\epsilon_2$ . Therefore, minimizing the sensitivity is possible by setting  $\epsilon_1$  as in Equation 14.

You can reduce the sensitivity of the BES with respect to the stopband rejection  $\epsilon_2$  by making  $\epsilon_2 \gg \epsilon_1$  for any value of  $\epsilon_1$ . Alternatively, you can reduce this sensitivity by making  $\epsilon_1$  small. This interaction of parameters is unique to elliptic filters.

Note from Equation 13 that the sensitivity of the stopband corner frequency  $\Omega_s$  increases as you decrease  $\Omega_s$ . However, decreasing  $\Omega_s$  increases the BES. Thus, although you can increase BES by reducing  $\Omega_s$ , you must temper your intent by the resulting increase in sensitivity. Consider the effective change in the BES along with the change in the associated sensitivity. Again, using the assumption that  $m' \rightarrow 0$ , you can rewrite Equation 4 as:

$$\text{BES} = A \left( \frac{\Omega_s^2}{\Omega_s^2 - 1} \right) \quad (15)$$

Taking the derivative of Equation 15 with respect to  $\Omega_s$  gives the rate of change of the BES with respect to the parameter you are modifying for the maximization:

$$\frac{\partial \text{BES}}{\partial \Omega_s} = -2A\Omega_s^3 D(\Omega_s), \quad (16)$$

where  $D(\Omega_s) = 1/(\Omega_s^2 - 1)$  is the stopband-frequency factor. As for the sensitivity of Equation 13, you can easily calculate

$$\frac{\partial S_{\Omega_s}^{\text{BES}}}{\partial \Omega_s} = 4\Omega_s D(\Omega_s) \quad (17)$$

Because  $\Omega_s > 1$  and  $\Omega_s^3 > \Omega_s$ , you can improve the BES of the filter at a greater rate than you degrade the corresponding sensitivity (Reference 4).

Maximizing the filter involves solving for the incremental order of the elliptic filter. You can obtain the order of an elliptic filter from a filter nomograph (Reference 1) or calculate the order using the following equation (Reference 5).

$$n = \frac{K(m)K(1-m')}{K(m')K(1-m)} \quad (18)$$

In Equation 18, K is the complete elliptic integral of the first kind (Reference 6) as follows:

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k \sin^2 \theta}} d\theta \quad (19)$$

You can find tabulated results of the above integral in mathematical handbooks or easily calculate the results using software packages such as MathCAD (Mathsoft Inc, Cambridge, MA).

The result of Equation 18 is a real number, and you select the next highest integer, that is

$$n_i = \left\lceil \frac{K(m)K(1-m')}{K(m')K(1-m)} \right\rceil \quad (20)$$

where the subscript *i* denotes an integer. You can always select a higher order to satisfy an arbitrary selectivity requirement, but it is useful to maximize the selectivity with no increase in order. As already noted,  $\epsilon_1$  and  $\epsilon_2$  are fixed for most practical cases, and  $m' = 0$ . Thus, the parameter  $\Omega_s$  is the degree of freedom for maximizing the selectivity of the filter while assuring that  $n_i$  remains fixed.

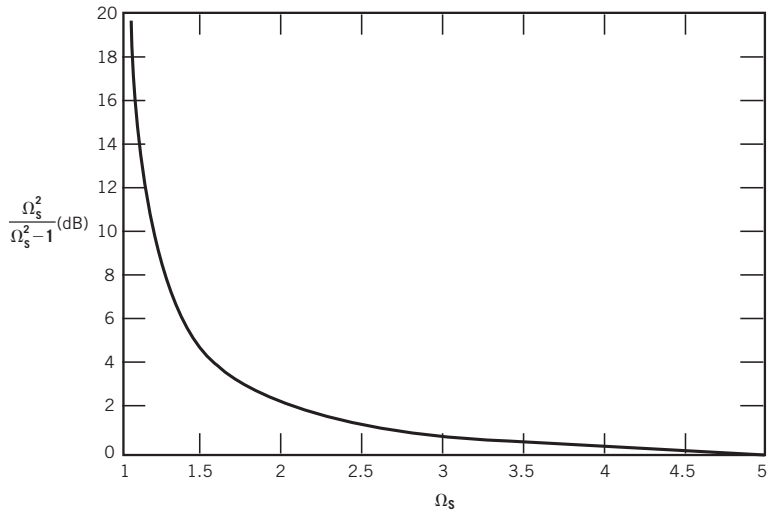
You now need to make a distinction between  $\tilde{\Omega}_s$ , which is the variable, and  $\Omega_s$ , which is the value of the specified stopband corner frequency. Because  $\tilde{\Omega}_s$  is the variable, you can write Equation 20 as

$$n_i = \left( \frac{K(m')}{K(1-m')} \right) \left( \frac{K(\tilde{m})}{K(1-\tilde{m})} \right) = C \left( \frac{K(\tilde{m})}{K(1-\tilde{m})} \right) \quad (21)$$

where C is a constant and  $\tilde{m} = 1/\tilde{\Omega}_s^2$ , such that

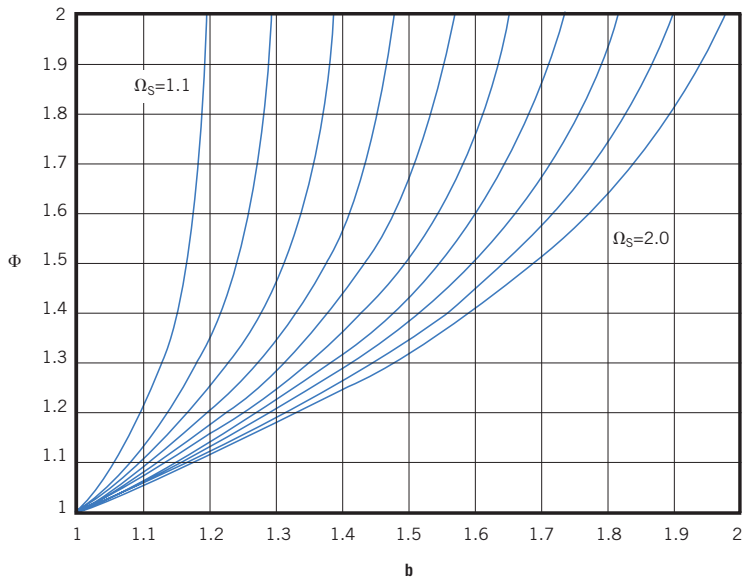
$$\frac{K(m)}{K(1-m)} \leq \frac{K(\tilde{m})}{K(1-\tilde{m})} \leq \frac{n_i}{n} \frac{K(m)}{K(1-m)} \quad (22)$$

Figure 2



A scaling factor determines the band-edge selectivity of an elliptic filter.

Figure 3



For a given  $V_s$  curve and value of  $\Phi$ , you can calculate  $\tilde{\Omega}_s = \Omega_s/b$ .

where  $n$  is the real number from the equality in Equation 18. Normalizing Equation 22 using  $K(m)/K(1-m)$  produces the result

$$1 \leq \frac{K(\tilde{m})K(1-m)}{K(m)K(1-\tilde{m})} \leq \frac{n_i}{n} \quad (23)$$

Because you must make  $\tilde{\Omega}_s < \Omega_s$  to in-

crease the selectivity, make

$$\tilde{\Omega}_s = \frac{\Omega_s}{b}, \quad b > 1 \quad (24)$$

Substituting Equation 24 into Equation 23 yields

$$1 \leq \Phi = \frac{K(b^2 m)K(1-m)}{K(m)K(1-b^2 m)} \quad (25)$$

This equation has the same form as the calculation of the filter order in **Equation 18**. Thus, you can use the same formulation and substitute the appropriate values. **Figure 3** shows a plot of  $\Phi$  versus  $b$  for various values of  $\Omega_s$ . To use this plot, follow four steps:

1. Calculate  $n$  from **Equation 18** and  $n_i$  from **Equation 21**.
2. Set  $\Phi_{\max} = n_i/n$  to set the “excess order.”
3. For the given  $\Omega_s$  curve, read  $b$  from the point where  $\Omega_s = \Phi_{\max} = n_i/n$ .
4. Calculate  $\tilde{\Omega}_s = \Omega_s/b$ .

This process results in the minimum stopband corner frequency  $\tilde{\Omega}_s$  that maximizes the BES for the given filter order. All other parameters remain fixed.

This technique can be useful with filter-design packages. Filter-design packages typically provide designs that meet the specifications but do not necessarily maximize the selectivity of the filter. A little extra work using the proposed technique results in a superior filter with no additional complexity. To use this technique with the filter software, you simply substitute the value  $\tilde{\Omega}_s$  for the original  $\Omega_s$  requirement for maximum selectivity.

**DESIGN EXAMPLE DEMONSTRATES TECHNIQUE**

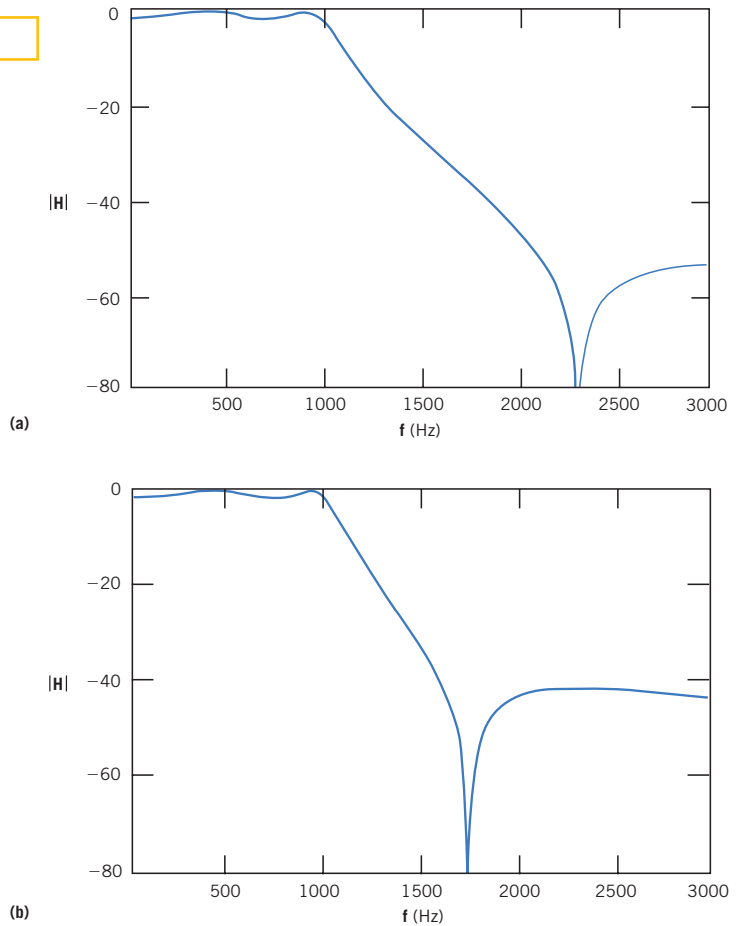
To demonstrate the effectiveness of the technique, consider the following lowpass-filter requirements: passband ripple  $M_p = 1.25$  dB, stopband rejection  $M_s = 40$  dB, passband frequency  $f_p = 1000$  Hz, and stopband frequency  $f_s = 2000$  Hz. From filter nomographs, you can quickly determine that this set of requirements would result in an eighth-order Butterworth or a fifth-order Chebyshev filter.

From  $M_p = 1.25$  dB,  $\epsilon_1 = 0.5775$ . From  $M_s = 40$  dB,  $\epsilon_2 = 100$ . From the frequency requirement,  $\Omega_s = f_s/f_p = 2$ . Using **Equation 18**, you can calculate the order  $n = 3.25482$ . Select the next highest order, so  $ni = 4$ .

**Figure 4a** shows the fourth-order elliptic filter that meets the requirements. For this filter, the BES calculated from **Equation 2** is 4.62. The sensitivities are  $S_{\epsilon_1}^{\text{BES}} = 1.25$ ,  $S_{\epsilon_2}^{\text{BES}} = 0$ , and  $S_{\Omega_s}^{\text{BES}} = -0.67$ .

To maximize the selectivity of the filter, you calculate  $\Phi_{\max} = n_i/n = 1.23$ . Reading this value from **Figure 3** at the  $\Omega_s = 2$  curve results in a value reading of

**Figure 4**



**A fourth-order elliptic filter (a) meets the following filter requirements: passband ripple = 1.25 dB, stopband rejection = 40 dB, passband frequency = 1000 Hz, and stopband frequency = 2000 Hz. Another fourth-order elliptic filter (b) not only meets the filter requirements but also exhibits maximum selectivity.**

$b = 1.35$ . Thus, the required stopband frequency  $\tilde{\Omega}_s = \Omega_s/b = 2/1.35 = 1.48$ . You use this value as the stopband corner frequency in the design and recalculate the filter poles and zeros. **Figure 4b** shows a plot of the fourth-order filter that meets the original requirements with maximum BES. The new BES is 6.36 with new sensitivity  $S_{\Omega_s}^{\text{BES}} = -1.68$ .

A careful observation of **Figure 4a** and **b** highlights the effect of moving in the stopband corner frequency  $\Omega_s$ . The original specifications resulted in a filter whose transition band just met the  $-40$ -dB rejection requirement at 2000 Hz (**Figure 4a**). The secondary lobe is down around  $-53$  dB with a notch at 2350 Hz. The proposed technique moved the notch closer to the passband to around

1750 Hz (**Figure 4b**). This notch movement results in an increase of the secondary lobe up to the required  $-40$ -dB rejection level. However, rejection in the transition band is superior. For example, the original filter had 25 dB of rejection at 1500 Hz. The modified filter has more than 30 dB of rejection at 1500 Hz.

**COMPARE RESPONSE TO CHEBYSHEV FILTER**

It is interesting to compare the elliptic filter to the Chebyshev filter, which like the elliptic filter provides selectivity that is proportional to  $n^2$ . A seventh-order Chebyshev filter is necessary to meet this new requirement of  $\Omega_s = 1.48$ . Therefore, the elliptic filter is the clear winner due to its reduced parts count in circuit implementations, even when fac-

toring in the transmission zeros.

Increasing filter selectivity has a negative impact on the delay response in the passband. Elliptic filters exhibit less delay variation than Chebyshev filters but more delay peaking. Negative delay impulses of area  $-\pi$  appear at the zero frequencies, and the effect of reducing  $\Omega_s$  simply moves the zero impulses closer to the transition band. However, to compensate for the zeros, the pole locations shift closer to the  $j\omega$  axis. This shift slightly increases delay variation but severely impacts delay peaking near the band edge. In addition, if the zeros are not purely imaginary but lay off the  $j\omega$  axis, they would produce negative delay

peaking of nonzero bandwidth, thereby distorting the delay near the passband edge.

Reducing  $\Omega_s$  also impacts the step response of elliptic filters. From the plots in **references 7 and 8**, the step response depends on the inverse of the stopband corner  $\Omega_s$  for constant in-band ripple. The low-frequency delay and thus the delay time decreases as  $\Omega_s$  decreases. In addition, the overshoot decreases as  $\Omega_s$  decreases. You can explain this fact by observing that the highest-Q complex-pole pair moves closer to the imaginary zeros as  $\Omega_s$  decreases, which reduces the residue value for that pole and, therefore, the overshoot.

For fixed  $\Omega_s$ , the overshoot increases with filter order. Therefore, maximizing selectivity not only reduces step-response overshoot but also ensures that there is no increase because the order remains fixed. Also, the rise time remains relatively constant as long as the 3-dB bandwidth is nearly constant, which is a characteristic of high-order filters.

Due to the increase in the filter's sensitivity to the stopband frequency ratio  $\Omega_s$ , for practical designs you should select a value for  $\tilde{\Omega}_s$  that is a little larger than the value that results in the maximum selectivity. A few Monte Carlo runs may be in order to evaluate the filter's sensitivity to the higher selectivity. □

## A NEW LOOK AT THE BRICK-WALL FILTER

The brick-wall filter in **Figure A** is valuable from a theoretical standpoint because it serves as the standard for filter approximations. However, you can never actually build this brick-wall filter for one simple reason: Its magnitude response is zero over an infinite band of frequencies. Aside from the fact that the phase is undefined for a band of zero frequencies, the finite bandwidth of the filter makes it noncausal, that is, it has a response without any input. Indeed, the impulse response of the brick-wall filter is the  $\sin x/x$  function, which extends for all time.

**Reference A** discusses the relation between causality, or realizability, and frequency response, deriving the following criterion where  $H(j\omega)$  is the frequency response of the filter:

$$\int_{-\infty}^{\infty} \frac{|\log |H(j\omega)||}{1 + \omega^2} d\omega < \infty.$$

This compact formula is difficult to evaluate except in the simplest cases, such as the brick-wall filter. If you let the passband of the brick-wall filter be unity and ex-

tend to frequency  $\omega_o$ , as **Figure A** shows, then you can break up the criterion as follows:

$$2 \int_0^{\infty} \frac{|\log |H(j\omega)||}{1 + \omega^2} d\omega = 2 \left[ |\log(1)| \int_0^{\omega_o} \frac{1}{1 + \omega^2} d\omega + |\log(0)| \int_{\omega_o}^{\infty} \frac{1}{1 + \omega^2} d\omega \right].$$

As in **Reference B**, the integral equals

$$\int_a^b \frac{1}{1 + \omega^2} d\omega = \tan^{-1} \omega \Big|_{\omega=a}^b.$$

Therefore,

$$2[|\log(1)| \tan^{-1}(\omega_o) + |\log(0)|(\pi/2 - \tan^{-1}(\omega_o))] = 0 + |\log(0)| = \infty.$$

This equation violates the Paley-Wiener criterion, and the brick-wall filter is not realizable.

It turns out that finite zeros in the magnitude response—like in the elliptic filter—are not a problem. Using the above argument, a single zero in the magnitude re-

sponse represents a phase or delay discontinuity, which itself is well-defined. Furthermore, the zero value of any number of magnitude components, as long as they are distinct, does not constitute a special problem in the causality of the impulse response. The Paley-Wiener criterion provides a test for a contiguous band of zeros.

This last point is a fundamental result of Fourier analysis. For example, the spectrum of a periodic waveform can have many discrete frequency terms, or harmonics, that are separated by bands of zero magnitude. Yet, a periodic waveform is not causal. On the other hand, the spectrum of a time-limited signal results in an infinite number of frequency components. Some of the components may be zero, but none are in a contiguous band.

Now look at the problem from the time domain. If a signal is causal, then its response is zero for time  $-\infty < t < t_o$ . Let's assume  $t_o = 0$  without any loss of generality. Then you can equivalently state causality as a signal gated, or multiplied, by a unit step at time  $t_o = 0$ . The Fourier transform of the unit step is (**Reference C**)

which consists of a function

$$\pi U_0(\omega) + \frac{1}{j\omega},$$

whose magnitude response varies as  $1/\omega$  and is centered at  $\omega = 0$  with a unit impulse,  $U_0$ , at  $\omega = 0$  (**Figure B**). Gating any input signal causes a convolution of the input spectrum with the spectrum in **Figure B**, resulting in frequency components in all bands because of the infinite spectrum of the unit step.

Does this situation mean that if the input is causal then you can have a noncausal impulse response and have a causal output? The answer is no. Recall that the impulse response of the filter convolves with the input, so if the impulse response is noncausal, the output is noncausal.

The discussion thus far shows, by way of the Paley-Wiener criterion and some basic results of Fourier analysis, that the brick-wall filter is not realizable. Now consider modifying the brick-wall filter by making the stopband magnitude response nonzero and a constant,  $\epsilon$  (**Figure C**). If you substitute this modified brick-wall filter into the Paley-Wiener criterion,



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BIOGRAPHY**

Celestino A Corral is a senior staff electrical engineer with the commercial, government, and industry solutions sector of Motorola Inc (Plantation, FL). He currently is the technical lead of the simulation and modeling engineering team, but his career has encompassed many aspects of analog and RF design as well as software development. He holds bachelor's, master's, and PhD degrees in electrical engineering from the University of Miami (Coral Gables, FL).



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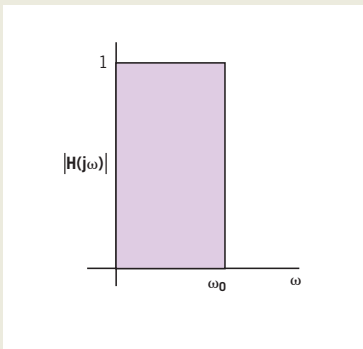
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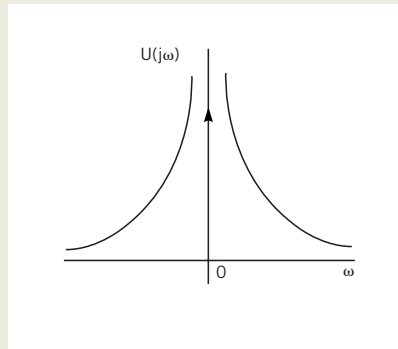
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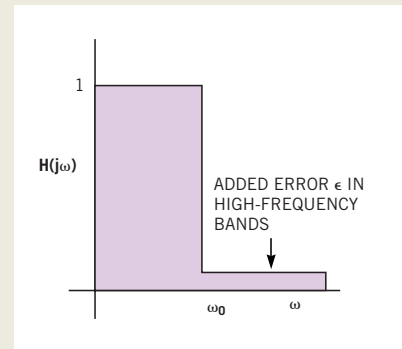
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**Figure A** You can't actually build this ideal brick-wall filter because the magnitude response is zero over an infinite band of frequencies.



**Figure B** The Fourier transform of a function whose magnitude response varies as  $1/\omega$  and is centered at  $\omega=0$  with a unit impulse  $U_0$  at  $\omega=0$ .



**Figure C** You can make a brick-wall filter realizable by making the stopband magnitude response nonzero and a constant  $\epsilon$ .

it is easy to show that the result is now finite (that is,  $|\log(\epsilon)|$  replaces  $|\log(0)|$  in the last term), and you have a causal filter.

This modification essentially adds a contiguous band of small magnitude and phase components from the stopband. These components contribute to the impulse response in such a fashion that the filter is now causal.

You can therefore choose to approximate the modified brick-wall filter instead of the ideal brick-wall filter. This approximation shifts attention from attempting to ap-

proximate something that you can't build to attempting to approximate something that you can. You can use finite zeros in the stopband to your advantage. Furthermore, any additional rejection in the stopband is acceptable. Elliptic filters excel in these requirements, which is why they are so useful in applications with stringent magnitude-response requirements.

Another interesting point is that the selectivity of the ideal and modified brick-wall filters is infinite. Because you can in theory

build a modified brick-wall filter, infinite selectivity is a worthy and achievable goal. From the derivations in this article (see **Equation 4** of the main text), the elliptic filter's selectivity is nearly infinite when  $\Omega_s \approx 1$ . Maximizing selectivity through the proposed method for higher elliptic-filter orders allows you to approximate the infinite selectivity of the ideal brick-wall filter response.

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